MATH21B – LECTURE 22: STABILITY
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1. Stability

Problem 1. Are the following matrices asymptotically stable?

(i) The matrix given by
$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}.$$

(ii) A (2×2) rotation matrix.

(iii) A dilation with scaling factor $0 < \rho < 1$.

(iv) A (2×2)-matrix with $\det(A) = 2$ and $\operatorname{tr}(A) = 3$.

(v) A^T when A is asymptotically stable.

(vi) A^5 when A is asymptotically stable.

Solution. (i) Its characteristic polynomial is $(1-\lambda)^2 - 1/4 = \lambda^2 - 2\lambda + 3/4$. This has roots $\frac{2 \pm \sqrt{4-3}}{2} = 1 \pm 1/2$, so it is not asymptotically stable.

(ii) It is asymptotically stable, any orthogonal matrix has eigenvalues satisfying $|\lambda_1 \cdots \lambda_n| = 1$, so that at least one satisfies $|\lambda_i| \geq 1$.

(iii) It is asymptotically stable, it has a single eigenvalue ρ.

(iv) We then have a characteristic polynomial given by $\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = \lambda^2 - 3\lambda + 1$ and thus roots $\frac{3 \pm \sqrt{9-4}}{2}$, the positive one being larger than 1.

(v) It is asymptotically stable because it has the same eigenvalues as A.

(vi) It is asymptotically stable because the eigenvalues of A^5 are the fifth powers of eigenvalues of A, and $|\lambda|^5 < 1$ is $|\lambda| < 1$. ∎

2. Recollection

Problem 2. True or false?

(i) The matrix $A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}$ is not asymptotically stable.

(ii) Every polynomial has a complex root.

(iii) If A is asymptotically stable and invertible, then A^{-1} is also asymptotically stable.

(iv) An orthogonal matrix has a real eigenvalue.

(v) We have that $\operatorname{tr}(A) = \operatorname{tr}(A^{-1})$.

(vi) A rotation matrix always has 1 as an eigenvalue.

(vii) A (2×2)-matrix A which satisfies $\operatorname{tr}(A) = 1$ and $\operatorname{tr}(A^2) = 2$ is diagonalizable.

(viii) To find a least squares solution of $A\vec{x} = \vec{b}$ you need to solve $A^T A\vec{x} = A^T\vec{b}$.

Solution. (i) True. It has eigenvalues given by $2 \pm \sqrt{2}i$, which have absolute value > 1.

(ii) True. This is a consequence of the fundamental theorem of algebra.

(iii) False, since the eigenvalues of A^{-1} are $1/\lambda_i$ for λ_i an eigenvalue of A, and $|1/\lambda_i| = 1/|\lambda_i|$, which is > 1 if $|\lambda_i| < 1$.

(iv) False, e.g. rotation by 90°.

(v) False, e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ has trace 4 but its inverse has trace 1.

(vi) False, e.g. rotation by 90°.

(vii) True. If $\text{tr}(A) = 1$ then $\lambda_1 + \lambda_2 = 1$. The eigenvalues of A^2 are λ_1^2 and λ_2^2, so we also have $\lambda_1^2 + \lambda_2^2 = 2$. In particular, λ_1 and λ_2 have to be distinct and so the matrix is diagonalizable.

(viii) True, that is finding a least squares solution is. ■

Summary

- A discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ is *asymptotically stable* if $\vec{x}(t) \to \vec{0}$ as $t \to \infty$. This is true if and only if all (complex) eigenvalues λ_i of A satisfy $|\lambda_i| < 1$.