MATH21B – LECTURE 21: COMPLEX EIGENVALUES
SPRING 2018, HARVARD UNIVERSITY

1. Complex numbers

Problem 1. (i) Compute $(a+bi)(a+bi)$.
(ii) Compute $(a+bi)(a-bi)$.
(iii) Take the square of both sides of Euler's identity $e^{i\theta} = \cos(\theta) + \sin(\theta)i$ to prove that $\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2$ and $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$.

Solution. (i) It is $a^2 - b^2 + 2abi$.
(ii) It is $a^2 + b^2$.
(iii) On the one hand we have $(e^{i\theta})^2 = e^{2i\theta} = \cos(2\theta) + \sin(2\theta)i$, while on the other hand we have $(\cos(\theta) + \sin(\theta)i)^2 = \cos(\theta)^2 - \sin(\theta)^2 + 2\cos(\theta)\sin(\theta)i$. Comparing the real and imaginary parts we prove the addition laws. ■

2. Complex eigenvalues

Problem 2. (i) What is the geometric interpretation of the following matrix?

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(ii) Explain geometrically why it has no real eigenvalues.
(iii) Find the complex eigenvalues and complex eigenvectors of A.

Solution. (i) It is a counterclockwise rotation by 90°.
(ii) It has no real eigenvalues because there is no vector with real entries which is mapped to a scaled version of itself under rotation by 90°.
(iii) The characteristic polynomial is $\lambda^2 + 1$. This has complex roots $\pm i$, so these are the complex eigenvalues. To compute the eigenvectors we must find vectors spanning the kernels of

$$A - i\mathrm{id}_2 = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \text{ and } A + i\mathrm{id}_2 = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}.$$

These are given by $[-i, 1]^T$ and $[i, 1]^T$. ■

Problem 3. Consider the (5×5)-matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(i) Find the characteristic polynomial of A.
(ii) What are the eigenvalues of A?

(iii) Note that an eigenvector $\vec{v}$ for an eigenvalue λ of A satisfies

$$A\vec{v} = \begin{bmatrix} v_5 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \lambda\vec{v}.$$

Find an eigenbasis for A.

Solution. (i) It is given by $\lambda^5 - 1 = 0$.

(ii) They are the five *roots of unity* $1, \xi \coloneqq \exp(2\pi i/5), \xi^2, \xi^3, \xi^4$.

(iii) The eigenbasis is given by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \xi^4 \\ \xi^3 \\ \xi^2 \\ \xi \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \xi^3 \\ \xi \\ \xi^4 \\ \xi^2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \xi^2 \\ \xi^4 \\ \xi \\ \xi^3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \xi \\ \xi^2 \\ \xi^3 \\ \xi^4 \end{bmatrix}.$$

■

Summary

- Complex numbers $z \in \mathbb{C}$ are numbers of the form $z = a + bi$ where $i = \sqrt{-1}$. You can picture these as points in the plane. Euler's formula says that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, which gives $e^{\pi i} + 1 = 0$ for $\theta = \pi$. The absolute value $|a + bi|$ is given by $\sqrt{a^2 + b^2}$.
- Every polynomial of degree n has n complex roots (counted with algebraic multiplicity). This implies that every matrix has a complex eigenvector, though it is still might not have an eigenbasis.