MATH21B – LECTURE 20: SIMILARITY AND DIAGONALIZATION
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1. Similarity and diagonalization

Problem 1. (i) Explain why the 0-eigenspace of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is 3-dimensional.

(ii) Use the trace to find the fourth eigenvalue.

(iii) If A is diagonalizable, what is the diagonal matrix B it is similar to?

(iv) Diagonalize A, that is, find S such that SAS^{-1} is diagonal.

Solution. (i) The rank is obviously 1, so the kernel has dimension 3. The kernel by definition coincides with the 0-eigenspace.

(ii) The matrix has eigenvalues 4 and 0. Necessarily 4 has geometric multiplicity 1. Thus the diagonal matrix is

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

or a matrix obtained by permuting the diagonal entries.

(iii) An eigenvector for the eigenvalue 4 is given by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The 0-eigenvectors are found by taking a basis for the kernel:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus we have

$$S = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

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Problem 2. In which of the following cases are the two matrices similar?

(i)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}.$$

(ii)
$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

(iii)
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

(iv)
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

(v)
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}.$$

Solution. (i) No, their traces are different.
(ii) No, the first has eigenvalue 2 with geometric multiplicity 1, while the second has eigenvalue 2 with geometric multiplicity 2.
(iii) Yes, both are diagonalizable and have the same eigenvalues.
(iv) No, their determinants are different.
(v) Yes, both are diagonalizable and have the same eigenvalues. ∎

2. Recollection

Problem 3. True or false?

(i) If λ is an eigenvalue of A then $-\lambda$ is an eigenvalue of $-A$.
(ii) There always exists a least squares solution $\vec{x}^*$ to a system of equations $A\vec{x} = \vec{b}$.
(iii) If A is an $(n \times n)$-matrix, then $\operatorname{rank}(A) + \operatorname{rank}(\mathrm{id}_n - A) = n$.
(iv) A reflection of $\mathbb{R}^3$ in a line is not similar to a rotation of $\mathbb{R}^3$ around an axis.
(v) A rotation dilation is always orthogonal.
(vi) If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of an orthogonal (3×3)-matrix A (counted with algebraic multiplicity), then $\lambda_1\lambda_2\lambda_3 = \pm 1$.
(vii) A (2×2)-matrix always has an eigenvector.

Solution. (i) True.
(ii) True.
(iii) False, e.g.
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
(iv) True, look at determinant.
(v) False, it can scale so that $\det \neq \pm 1$.
(vi) True, $1 = \det(AA^T) = \det(A)^2 = (\lambda_1\lambda_2\lambda_3)^2$.
(vii) False, e.g.
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$
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Summary

- Recall that $(n \times n)$-matrices A and B are *similar* if there is an invertible matrix S such that $B = S^{-1}AS$ (i.e. they differ by a change of basis). Similar matrices have the same determinant, same trace, same eigenvalues, same characteristic polynomial, same algebraic multiplicities, and same geometric multiplicities. The converse is not true.
- An $(n \times n)$-matrix A is *diagonalizable* if it is similar to a diagonal matrix B. Then in $B = S^{-1}AS$, the columns of S are an eigenbasis for A, and we diagonalize a matrix by finding an eigenbasis. Thus if A has n different eigenvalues it is diagonalizable.