MATH21B – LECTURE 2: GAUSS-JORDAN ELIMINATION
SPRING 2018, HARVARD UNIVERSITY

1. Gauss-Jordan elimination

Problem 1. (i) Which of the following matrices is in reduced row echelon form?

$$\text{(a)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{(b)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{(c)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

(ii) For those of the above matrices that are in reduced row echelon form, give the number of leading 1's.

Solution. (i) Only (a).
(ii) It has a single leading 1. ■

Problem 2. (i) Use Gauss-Jordan elimination to find the reduced row echelon form of the following augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 3 \\ 3 & 3 & 3 & 1 \\ 3 & 3 & 4 & 0 \end{array}\right].$$

(ii) What are the solutions of the corresponding system of linear equations?

Solution. (i) We get

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

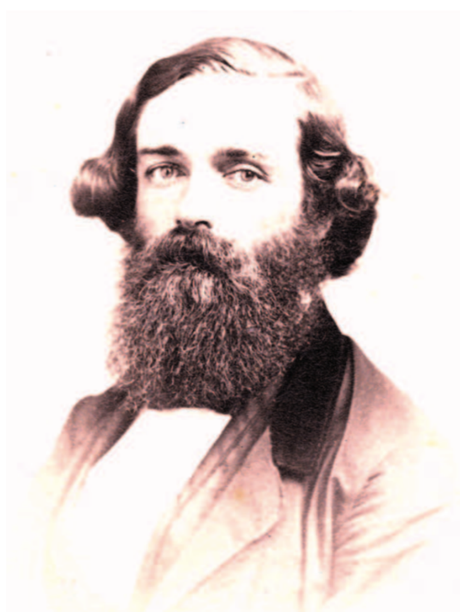

Figure 1. Myrick Doolittle, from Gcrar, *Mathematicians of Gaussian Elimination.*

(ii) The last row means $0 = 1$, which is impossible. This means that there is no solution: the system of linear equations is *inconsistent*. ■

2. Leading 1's, free variables, and rank

Problem 3. (i) Find all 3×2-matrices in reduced row echelon form which have two leading 1's. What are their ranks? What are the free variables of the corresponding system of linear equations?

(ii) Find all 2×3-matrices in reduced row echelon form which have two leading 1's. What are their ranks? What are the free variables of the corresponding system of linear equations?

Solution. (i) There is only

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This has rank 2, as in the problem we demanded it has two leading 1's. There are no free variables.

(ii) There are three cases

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix} \quad \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $a, b \in \mathbb{R}$ may be chosen arbitrarily. They all have rank 2, as in the problem we demanded it has two leading 1's. The first as x_3 as a free variable, the second x_2, and the third has no free variables. ■

Problem 4. (i) What is the rank of the following matrix?

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 3 \\ 5 & 3 & 5 \\ -6 & 5 & -6 \end{bmatrix}.$$

(ii) What are the free variables of the corresponding system of linear equations?

(iii) Show that the system of linear equations corresponding to the following augmented matrix is inconsistent:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 3 & 3 & 3 & 3 \\ 5 & 3 & 5 & 5 \\ -6 & 5 & -6 & 5 \end{array}\right].$$

(iv) (Challenge) Is it possible to change the 5 in the right-bottom corner to a different number so that the system becomes consistent? If so, which numbers work?

Solution. (i) The reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This has two leading 1's so the rank is 2.

(ii) The free variable is x_3.

(iii) The reduced row echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This has a leading 1 in the last column, so is inconsistent (due to the third row expressing the equation $0 = 1$).

(iv) Yes, it needs to be -6.

■

Summary

- We can write a system of linear equations as an augmented matrix $[A \mid \vec{b}]$.
- In *Gauss-Jordan elimination* we simplify the (augmented) matrix using the following three operations (the three S's)
  - (I) Swap two rows.
  - (II) Scale a row by a non-zero number.
  - (III) Substract a multiple of a row from another.
- During Gauss-Jordan elimination, our goal is to put the matrix in *reduced row echelon form*:
  - In a non-zero row, the first non-zero entry is 1.
  - If a column has a leading 1, all the other entries in the column are 0.
  - In a row with a leading 1, every row above has a leading 1 to the left.

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & \underset{\substack{\uparrow \\ \text{free variable}}}{0} & 0 & 0 \end{bmatrix}$$

- The *rank* of A is the number of leading 1's in $\text{rref}(A)$.