MATH21B – LECTURE 19: EIGENSPACES
SPRING 2018, HARVARD UNIVERSITY

1. The trace

Problem 1. The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

has two eigenvalues $5 \pm \sqrt{21}$. Use the trace to compute the third eigenvalue.

2. Geometric multiplicity and eigenbases

Problem 2. Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

(i) A has a single eigenvalue. What is it and what is its algebraic multiplicity?
(ii) What is its geometric multiplicity?

Solution. (i) The single eigenvalue is 3 and its algebraic multiplicity is 5 (whenever you have an upper- or lower-triangular matrix you can read off the eigenvalues and their algebraic multiplicities from the diagonal).
(ii) The 3-eigenspace is the kernel of

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which has rank 4, so that the kernel has dimension 1. Thus the geometric multiplicity is 1. ■

Problem 3. (i) Find an eigenbasis for the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}.$$

(ii) Does the following matrix have an eigenbasis?

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

Solution. Since the trace is 8, so the remaining eigenvalue is -2. ■

Solution. (i) The eigenvalues are 1 and 3, and eigenvectors for these eigenvalues are given by

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

(ii) No, its only eigenvalue is 3 but its geometric multiplicity is 1.

■

3. Recollection

Problem 4. True or false?

(a) $\text{tr}(A) = \text{tr}(A^T)$.
(b) A reflection of $\mathbb{R}^3$ in a plane has eigenvalues ± 1.
(c) The kernel of a non-invertible matrix is its 0-eigenspace.
(d) A matrix can have infinitely many eigenvectors.
(e) A matrix can have infinitely many eigenvalues.
(f) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.
(g) If the only eigenvalue of a matrix A is 1, then A is the identity matrix.
(h) If the only eigenvalue of a (4×4)-matrix A is 0 and its geometric multiplicity is 4, then A is the zero matrix.
(i) Every matrix has an eigenvector.
(j) (Hard) Every $(n \times n)$-matrix for n odd has an eigenvector.

Solution. (a) True, as the diagonal entries of A and A^T are the same.
(b) True, geometrically we see the plane of reflection is the 1-eigenspace while the line orthogonal to the plane of reflection is the -1-eigenspace.
(c) True, by definition.
(d) True, every non-zero multiple of an eigenvector is again an eigenvector.
(e) False, a non-zero polynomial has only finitely many roots.
(f) True, as the diagonal entries of $A + B$ are the sums of the diagonal entries of A and B.
(g) False, e.g.
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$
(h) True, as the geometric multiplicity being 4 implies that all of $\mathbb{R}^4$ is the 0-eigenspace = the kernel.
(i) False, e.g. the rotation matrix
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$
(j) True, as $\det(A - \lambda \text{id}_n)$ is an odd degree polynomial and thus has a real root. ■

Summary

- The coefficient of λ^{n-1} in the characteristic polynomial is $(-1)^{n-1}\text{tr}(A)$, where the *trace* $\text{tr}(A)$ is the sum of the diagonal entries. The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomials and the trace $\text{tr}(A)$ is also equal to the sum of eigenvalues counted with algebraic multiplicity.
- If λ is an eigenvalue of A, then $E_\lambda = \ker(A - \lambda \text{id}_n)$ is the λ-*eigenspace* of A. Its dimension is called the *geometric multiplicity* of λ. We have that $1 \leq$ geometric multiplicity $\leq$ algebraic multiplicitivity.
- If an $(n \times n)$-matrix A has n distinct eigenvalues, then there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A, which is called an *eigenbasis*.