MATH21B – LECTURE 18: EIGENVALUES, EIGENVECTORS AND DISCRETE DYNAMICAL SYSTEMS
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1. Fibonacci numbers

In Section 3 of *Liber Abbaci*, Leonardo of Pisa (c. 1175 – c. 1250, later known as Fibonacci) poses the following question:

> A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

Problem 1. Compute the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution. The characteristic polynomial is given by

$$\det(A - \lambda \mathrm{id}_2) = \lambda^2 - \lambda - 1.$$

Its roots are $\frac{1 \pm \sqrt{5}}{2}$, so these are the eigenvalues. To find the eigenvectors we need to find the kernels of

$$\begin{bmatrix} 1/2 - \sqrt{5}/2 & 1 \\ 1 & -1/2 - \sqrt{5}/2 \end{bmatrix}, \quad \begin{bmatrix} 1/2 + \sqrt{5}/2 & 1 \\ 1 & -1/2 + \sqrt{5}/2 \end{bmatrix}.$$

These are spanned by the following two vectors respectively

$$\begin{bmatrix} -1 \\ 1/2 - \sqrt{5}/2 \end{bmatrix}, \qquad \begin{bmatrix} -1 \\ 1/2 + \sqrt{5}/2 \end{bmatrix}.$$ ■

2. The characteristic polynomial

Problem 2. (i) Compute the determinants of the following matrices:

$$\text{(a)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(ii) Compute the characteristic polynomials and eigenvalues of (a), (b) and (c).
(iii) What are the eigenvectors for the different eigenvalues of (a), (b) and (c)?
(iv) Can you find a geometric explanation for the eigenvalues and eigenvectors of (c)?

Solution. (i) They are given by: (a) det $= 1$, (b) det $= 0$, (c) det $= 1$.
(ii) Their characteristic polynomials are given by: (a) $1 - 2\lambda + \lambda^2$, (b) λ^2, (c) $1 + \lambda - \lambda^2 - \lambda^3$. The eigenvalues are (a) 1 with algebraic multiplicity 2, (b) 0 with algebraic multiplicity 2, (c) -1 with algebraic multiplicity 2 and 1 with algebraic multiplicity 1.
(iii) For (a) every non-zero vector is an eigenvector, for (b) it is all non-zero multiples of $\vec{e}_1$, (c) non-zero vectors in $\mathrm{span}(\vec{e}_1, \vec{e}_2)$ are eigenvectors with eigenvalue -1, and the non-zero multiples of $\vec{e}_3$ are eigenvectors with eigenvalue 1. ■

Summary

- If $\vec{v}$ is a non-zero vector such that $A\vec{v} = \lambda\vec{v}$, then $\vec{v}$ is called an *eigenvector* of A and λ is its *eigenvalue*. The eigenvalues are the roots of the characteristic polynomial $\det(A - \lambda \mathrm{id}_n)$, whose constant coefficient is $\det(A)$. The eigenvectors for eigenvalue λ_i are obtained by finding $\ker(A - \lambda_i \mathrm{id}_n)$.
- If $\vec{v} \mapsto A\vec{v}$ describes the change in state after one time step of a discrete dynamical, then we can find the trajectories $A^s\vec{v}$ by decomposing $\vec{v}$ into eigenvectors; if $\vec{v} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$ with $\vec{v}_i$ an eigenvector of A with eigenvalue λ_i, then

$$A^s\vec{v} = c_1\lambda_1^s\vec{v}_1 + \ldots + c_n\lambda_n^s\vec{v}_n.$$