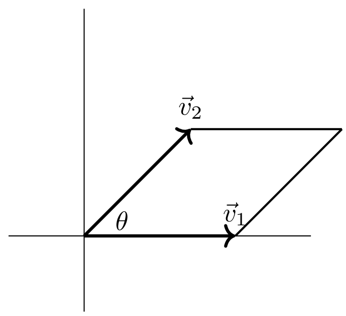
MATH21B – LECTURE 17: DETERMINANTS II
SPRING 2018, HARVARD UNIVERSITY

1. Geometry of the determinant

Problem 1. For which angles θ does the parallelogram with sides given by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

have the largest area?



Solution. The parallellogram is the image of the standard square under

$$\begin{bmatrix} 1 & \sin(\theta) \\ 0 & \cos(\theta) \end{bmatrix}.$$

Thus its area is $|\sin(\theta)|$, which has a maximal value of 1 at $\theta = 90°, 270°$. ■

2. Properties of the determinant

Problem 2. Compute $\det(A^{2018})$ where

$$A = \begin{bmatrix} 4 & 13 \\ 1 & 3 \end{bmatrix}.$$

Solution. We have that $\det(A) = -1$, so $\det(A^{2018}) = \det(A)^{2018} = (-1)^{2018} = 1$. ■

Problem 3. True or false?

(i) You can only take the determinant of a square matrix.
(ii) If a matrix has integer entries then the determinant is an integer.
(iii) If the determinant is an integer then matrix has integer entries.
(iv) A matrix with positive entries has positive determinant.
(v) $\det(A + B) = \det(A) + \det(B)$.
(vi) A rotation has determinant 1.
(vii) A reflection has determinant 1.
(viii) A projection has determinant 1.
(ix) If A is a (5×5)-matrix satisfying $A = -A^T$ then $\det(A) = 0$.

(x) The following matrix is invertible:

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}.$$

Solution. (i) True, by definition.

(ii) True, the Leibniz definition shows its a sum of powers of integers.

(iii) No, e.g.

$$\det \begin{bmatrix} \pi & \pi \\ \pi & \pi \end{bmatrix} = 0.$$

(iv) No, e.g.

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

(v) No, e.g. $\det(\mathrm{id}_2 - \mathrm{id}_2) = 0$ while $\det(\mathrm{id}_2) = \det(-\mathrm{id}_2) = 1$.

(vi) Yes.

(vii) No, only if its reflection about a k-dimensional subspace and $n - k$ is even.

(viii) No, it will often non-invertible and hence have determinant 0.

(ix) Yes, $\det(A) = \det(-A^T) = (-1)^5 \det(A^T) = -\det(A)$ so $\det(A) = 0$.

(x) Yes, its determinant is $4 \cdot 9 = 36 \neq 0$. ■.

Summary

- Geometrically, $|\det(A)|$ is the volume of the image of the unit cube under the linear transformation corresponding to A. If the sign $\det(A)$ is positive A preserves the orientation, if it is negative A flips the orientation.
- Here are the important properties of the determinant: (i) $\det(AB) = \det(A)\det(B)$, (ii) $\det(A^T) = \det(A)$, (iii) $\det(A^{-1}) = 1/\det(A)$, (iv) $\det(SAS^{-1}) = \det(A)$, (v) $\det(A^n) = \det(A)^n$.
- Let A_{ij} be the matrix obtained by A by deleting the ith row and jth column. Then we have
$$(A^{-1})_{ij} = (-1)^{i+j}\det(A_{ji})/\det(A).$$
- Questions you might ask yourself before computing a determinant $\det(A)$?
  - Is A upper or lower triangular?
  - Is A a block sum of two smaller matrices?
  - Is A non-invertible (so that $\det(A) = 0$)?
  - Is A a product of matrices?
  - Are there only a few non-zero patterns?
  - Is there a row or column with lots of zeroes to start Laplace expansion?
  - Can I swap or add rows or columns to simplify A?
  - Can I quickly row reduce to the upper-triangular case?
  - (Later: can we compute the eigenvalues of A?)