MATH21B – LECTURE 16: DETERMINANTS I
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1. Computing determinants using Leibniz's definition

Problem 1. Compute

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Solution. $\det(A) = aei - afh - bdi + bfg + cdh - ceg$. ■

Problem 2. Compute

$$\text{(a) } \det \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{(b) } \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}, \quad \text{(c) } \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 6 & 8 & 0 \\ 4 & 7 & 9 & 10 \end{bmatrix}.$$

Solution. They are given by 1, 400, 400. In all cases only one pattern is non-zero. ■

2. Computing determinants using other methods

Problem 3. Compute

$$\det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 5 \\ 0 & 7 & 3 & 3 \\ 0 & 0 & -10 & 2 \end{bmatrix}.$$

Solution. We need to do 3 row switches to put this is in upper diagonal form, so it is $(-1)^3 \cdot -70 = 70$. ■

Problem 4. Compute

$$\det \begin{bmatrix} 3 & 2 & 1 & 3 & 4 & 0 & 0 & 0 \\ 0 & 2 & 5 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

by using either (a) thinking of it as a block sum, (b) Laplace expansion, or (c) Gauss-Jordan elimination.

Solution. I would probably use (a) and then note the matrices are upper and lower diagonal. You get that it is $48 \cdot 1 = 48$. ■

Summary

- Leibniz's definition of the *determinants* $\det(A)$ of an $(n \times n)$-matrix is given by
$$\det(A) = \sum_{\pi} \operatorname{sign}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}$$
where $a_{i\pi(i)}$ denotes the entry in the ith row and $\pi(i)$th column.
- You can compute it recursively by *Laplace expansion*. Let A_{ij} be the matrix obtained by A by deleting the ith row and jth column. Then we have
$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(A_{1j})$$
where a_{1j} are the entries of A on the first row. You can pick any other row or column to expand, as long as you introduce the appropriate sign.
- If a matrix is block sum of two smaller matrices:
$$M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$
then $\det(M) = \det(A)\det(B)$.
- You can obtain the determinant using Gauss-Jordan elimination. If A is a matrix and during Gauss-Jordan elimination you scale by factors $\lambda_1, \dots, \lambda_k$ and swap rows s times, then
$$\det(A) = (-1)^s \frac{1}{\lambda_1 \cdots \lambda_k} \det(\operatorname{rref}(A)).$$
In particular $\det(A) \neq 0$ if and only if A is invertible.