MATH21B – LECTURE 14: ORTHOGONAL TRANSFORMATIONS
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1. The transpose matrix and dot products

Problem 1. A matrix A is *symmetric* is $A = A^T$.

(i) Is the following matrix symmetric?

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix}$$

(ii) Give an example of a symmetric (3×3)-matrix.

(iii) If A and B are symmetric $(n \times n)$-matrices, is AB necessarily symmetric? If yes, explain why. If no, give a counter-example.

Solution. (i) No.

(ii) The identity matrix.

(iii) No, as $(AB)^T = B^T A^T = BA$. We thus need to find two matrices which do not commute. Here is a random example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

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2. Orthogonal transformations

Problem 2. (i) Which of the following matrices are orthogonal?

$$\text{(a)} \begin{bmatrix} \cos(2) & -\sin(2) & 0 \\ \sin(2) & \cos(2) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{(d)}\, 1/3 \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix},$$

(e) the matrix for a rotation, (f) the matrix for an orthogonal projection onto a proper subspace.

(ii) Compute the inverses of those matrices in (i)(a)–(i)(d) which are orthogonal.

Solution. (i) The matrices (a), (c), (d) and (e) are, the first three verified by checking that the transpose is the inverse and (e) was explained in lecture. The matrix (b) can't be orthogonal since it is not invertible, and the same is the case (f).

(ii) Just take the transpose of (a), (c) and (d).

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Problem 3. Assuming A is orthogonal, which of the following matrices are also orthogonal?

(a) A^T.
(b) A^{-1}.
(c) A^2.
(d) $A^T A$.
(e) $A + A$.
(f) AA^{-1}.
(g) SAS^T for S orthogonal.

(h) SAS^{-1} for S invertible.

Solution. (a) Yes. We need to check that $(A^T)^T A^T = \text{id}$, but $(A^T)^T = A$ and $A^T = A^{-1}$ so this is true.
(b) Yes, $A^T = A^{-1}$, so use (b).
(c) Yes, $(A^2)^T = (A^T)^2 = (A^{-1})^2 = (A^2)^{-1}$.
(d) Yes, $(A^T A)^T = A^T (A^T)^T = A^T A$. (Note we didn't even use that A was orthogonal).
(e) No, as $(A + A)^T (A + A) = 2A^{-1} 2A = 4\text{id}$.
(f) Yes, $AA^{-1} = \text{id}$, which is orthogonal.
(g) Yes, $(SAS^T)^T = (S^T)^T A^T S^T = SA^{-1}S^T$ and we have that $(SA^{-1}S^T)(SAS^T) = SAA^{-1}S^T = SS^T = \text{id}$ as $S^T S = \text{id}$ as S is orthogonal.
(h) No, e.g. if S is a shear. ∎

Summary

- The transpose A^T is obtained by switching columns and rows. It satisfies $(A + B)^T = A^T + B^T$ and $(\lambda A)^T = \lambda A^T$. Slightly harder to see is that it satifies $(AB)^T = B^T A^T$ (so *switches* order of multiplication!), and $(A^{-1})^T = (A^T)^{-1}$).
- An $(n \times n)$-matrix is *orthogonal* if $A^T A = \text{id}_n$. This is equivalent to the columns of A being an orthonormal basis of $\mathbb{R}^n$. In particular, in a QR-decomposition of an $(n \times n)$-matrix the Q is orthogonal.
- The inverse of an orthogonal matrix A is A^T. The product of two orthogonal matrices is orthogonal.
- Orthogonal matrices preserve angles and length. Examples are rotations and reflections.