MATH21B – LECTURE 13: GRAM-SCHMIDT AND QR-DECOMPOSITIONS SPRING 2018, HARVARD UNIVERSITY

1. Gram-Schmidt

Problem 1. Consider the linear subspace V of $\mathbb{R}^3$ given by the span of

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(i) Find an orthonormal basis for V.
(ii) Find the matrix for orthogonal projection onto V.

Solution. (i) We start by making $\vec{v}_1$ have length 1 by dividing it by $||\vec{v}_1|| = \sqrt{2}$. Call the result $\vec{w}_1$:

$$\vec{w}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

Next we make the dot product between $\vec{w}_1$ and $\vec{v}_2$ equal to 0: do this we replace $\vec{v}_2$ by $\vec{w}_2'$ given by

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 1/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1 \end{bmatrix}$$

The vector $\vec{w}_2'$ has dot product with $\vec{w}_1$ given by

$$\vec{v}_2 \cdot \vec{w}_1 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1 \cdot \vec{w}_1 = \vec{v}_2 \cdot \vec{w}_1 - \vec{v}_2 \cdot \vec{w}_1 = 0$$

since $\vec{w}_1 \cdot \vec{w}_1 = ||\vec{w}_1||^2 = 1$. It has length $3/2$, so we divide by $\sqrt{3/2}$ to get $\vec{w}_2$ given by

$$\vec{w}_2 = \begin{bmatrix} \sqrt{1/6} \\ -\sqrt{1/6} \\ -\sqrt{2/3} \end{bmatrix}.$$

The vector $\vec{w}_1$ and $\vec{w}_2$ are the desired basis.

(ii) We need to take QQ^T with Q the matrix with columns $\vec{w}_1$, $\vec{w}_2$. ■

Problem 2. Apply the Gram-Schmidt procedure to

$$\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$$

to find an orthonormal basis of $\mathbb{R}^3$.

Solution. Call the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$. We first make $\vec{v}_1$ have length 1 by dividing by 2.

$$\vec{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Next we make the dot product of $\vec{v}_2$ with $\vec{w}_1$ equal to 0 by subtracting $(\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1$ to get $(2,0,0)^T$ and dividing by 2 to get the vector $\vec{w}_2$ of length 1 given by

$$\vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then we make the dot products of $\vec{v}_3$ with $\vec{w}_1$ and $\vec{w}_2$ equal to 0 by subtracting $(\vec{w}_1 \cdot \vec{v}_3)\vec{w}_1 + (\vec{w}_2 \cdot \vec{v}_3)\vec{w}_2$ to get $(0,0,4)^T$ and divide by 4 to get

$$\vec{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

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2. QR-decompositions

Problem 3. Find the QR-decomposition of

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Solution. We need to apply Gram-Schmidt and record the coefficients. The first column already has length 1, so we may take $\vec{v}_1 = \vec{w}_1$. Next we make the dot product of the second column with the first column equal to 0 by adding $\vec{w}_1$ to get $\vec{w}_2 = (1,0)^T$, which already has length 1. Thus we have expressions

$$\vec{v}_1 = \vec{w}_1$$
$$\vec{v}_2 = \vec{w}_2 - \vec{w}_1$$

From this we get the QR-decomposition

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

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Problem 4. Find the QR-decomposition of the matrix

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution. In problem 2 we found the orthonormal basis $\vec{w}_1$, $\vec{w}_2$ and $\vec{w}_3$ obtained by applying Gram-Schmidt to the columns. During this procedure we found

$$\vec{v}_1 = 2\vec{w}_1$$
$$\vec{v}_2 = 2\vec{w}_2 + 3\vec{w}_1$$
$$\vec{v}_3 = 4\vec{w}_3 + 4\vec{w}_1 + \vec{w}_2$$

From this we get the QR-decomposition

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

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Summary

- The Gram-Schmidt procedure produces an orthonormal basis $\{\vec{w}_1, \ldots, \vec{w}_k\}$ from a basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$. It is given by iteratively scaling and making the dot products 0:

$$\vec{w}_1 = \frac{1}{||\vec{v}_1||}\vec{v}_1$$

$$\vec{w}_2 = \frac{1}{||\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1||}(\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1)$$

$$\vdots$$

$$\vec{w}_k = \frac{1}{||\vec{v}_k - (\sum_{i=1}^{k-1}(\vec{v}_k \cdot \vec{w}_i)\vec{w}_i)||}\left(\vec{v}_k - (\sum_{i=1}^{k-1}(\vec{v}_k \cdot \vec{w}_i)\vec{w}_i)\right)$$

- If A is the matrix with columns given by the $\vec{v}_i$'s, then we have $A = QR$ with Q a matrix with orthonormal basis for columns, given by the $\vec{w}_i$'s,

$$Q = \begin{bmatrix} \vdots & \vdots & \cdots \\ \vec{w}_1 & \vec{w}_2 & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix}$$

  and R an upper triangular matrix given by expressing the $\vec{v}_i$'s in terms of the $\vec{w}_i$'s

$$\vec{v}_i = r_{ii}\vec{w}_i + \sum_{j<i} a_{ji}\vec{w}_j$$

  and taking the matrix

$$R = \begin{bmatrix} r_{11} & a_{12} & a_{13} & \cdots \\ 0 & r_{22} & a_{23} & \cdots \\ 0 & 0 & r_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

  This is called the QR-decomposition of A, and it is useful for computations.