MATH21B – LECTURE 12: ORTHOGONALITY
SPRING 2018, HARVARD UNIVERSITY

1. Dot products, orthogonality, length and angles

Problem 1. Consider the vectors

$$\vec{a}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

(i) Compute their lengths.
(ii) Compute the angle between them.
(iii) Find a basis $\vec{v}_3$, $\vec{v}_4$ for the orthogonal complement of the span of $\vec{v}_1$, $\vec{v}_2$.
(iv) Can we find $\vec{v}_3$, $\vec{v}_4$ in (iii) such that $\vec{v}_1, \ldots, \vec{v}_4$ is an orthonormal basis of $\mathbb{R}^4$?
(v) Give the matrix for orthogonal projection onto $\text{span}(\vec{v}_1, \vec{v}_2)^\perp$.

Solution. (i) Their lengths are 1.
(ii) They are orthogonal, so the angle between them is 90°.
(iii) The orthogonal complement is the kernel of the matrix with rows $\vec{a}_1$, $\vec{a}_2$, so we need to compute

$$\text{rref}\left(\begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

We find a basis

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(iv) Yes, they are already orthogonal, so we let need to scale them to have length one by dividing by $\sqrt{2}$.
(v) We need to take

$$Q = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix},$$

so that

$$QQ^T = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}.$$ ■

Problem 2. Fix a vector $\vec{v} \in R^n$.

(i) What is the matrix A for the system of equations $(\vec{x} \cdot \vec{v})\vec{v} = \vec{0}$ with unknowns the entries of $\vec{x}$?
(ii) What is the image of A?
(iii) What is the kernel of A?
(iv) If $||\vec{v}|| = 1$, give an interpretation of the linear transformation $T(\vec{x}) = (\vec{x} \cdot \vec{v})\vec{v}$.

Solution. (i) The ith row of A is given by $v_i \cdot \vec{v}$.
(ii) It is the span of $\vec{v}$.

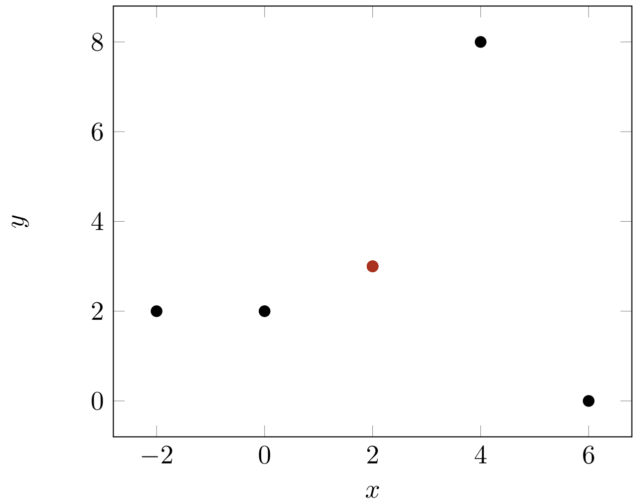

FIGURE 1. Part 3(ii).

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(iii) It is the orthogonal complement of the span of $\vec{v}$. ■

2. STATISTICS

Problem 3. Suppose we are given the data points $(0, 2)$, $(4, 8)$, $(6, 0)$ and $(-2, 2)$.

(i) Find the *expectation* $E[x]$ of the *random variable* given by taking the average of the four x-coordinates. Find the expectation $E(y)$.
(ii) Plot the four points and the point $(E[x], E[y])$.
(iii) Find the *deviation vectors* X and Y, obtained by taking as entries x-coord $- E[y]$, and y-coord $- E[y]$. These are also called the *centered random variables*.
(iv) Compute the *variance* of x and y, defined as $\text{Var}[x] = (X \cdot X)/4$, $\text{Var}[y] = (Y \cdot Y)/4$.
(v) Compute *standard deviations* $\sigma[x] = \sqrt{\text{Var}(x)}$, $\sigma[y] = \sqrt{\text{Var}(y)}$.
(vi) Compute the *covariance* $\text{Cov}[x, y] = (X \cdot Y)/4$. Show that the covariance is also equal to $E[(x - E[x])(y - E[y])]$ and to $E[xy] - E[x]E[y]$.
(vii) Compute the *correlation* $\text{Cor}[x, y] = (X \cdot Y)/(||X||||Y||)$.

Solution. (i) $E[x]$ is $(0 + 4 + 6 - 2)/4 = 2$ and $E[y]$ is $(2 + 8 + 0 + 2)/4 = 3$.
(ii) The point $(E[x], E[y])$ is red in Figure 1.
(iii) They are given by

$$X = \begin{bmatrix} -2 \\ 2 \\ 4 \\ -4 \end{bmatrix}, Y = \begin{bmatrix} -1 \\ 5 \\ -3 \\ -1 \end{bmatrix}.$$

(iv) They are $\text{Var}[x] = 10$, $\text{Var}[y] = 9$.
(v) They are $\sigma[x] = \sqrt{5}$, $\sigma[y] = 3$.
(vi) It is 1. By definition $E[(x - E[x])(y - E[y])]$ is computed by $(X \cdot Y)/4$. To write this, we use

$$\begin{aligned} E[(x - E[x])(y - E[y])] &= E[(xy - E[x]y - xE[y] + E[x]E[y]) \\ &= E[xy] - E[x]E[y] - E[x]E[y] + E[x]E[y] \\ &= E[xy] - E[x]E[y]. \end{aligned}$$

(vii) We get $\frac{1}{3\sqrt{10}}$. ■

Summary

- Two vectors $\vec{x}$, $\vec{y}$ are *orthogonal* if $\vec{x} \cdot \vec{y} = 0$. The *orthogonal complement* $V^{\perp}$ of a linear subspace V of $\mathbb{R}^n$ is given by $\{\vec{x} \mid \vec{x} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in V\}$. It is a linear subspace and its dimension $n - \dim(V)$.
- The *length* of a vector $\vec{x}$ is $||\vec{x}|| = \sqrt{\vec{x} \cdot \vec{x}}$. Pythagoras says that $||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$ if $\vec{x}$ and $\vec{y}$ are orthogonal. There are some standard inequalities:

  Cauchy-Schwartz inequality $|\vec{x} \cdot \vec{y}| \leq ||\vec{x}||||\vec{y}||$ (with equality if $\vec{x}$ and $\vec{y}$ are parallel),

  Triangle inequality: $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$.
- The *angle* α between two vectors is given by $\cos(\alpha) = (\vec{x} \cdot \vec{y})/(||\vec{x}||||\vec{y}||)$. Thus orthogonal vectors have angle 90° (or 270°) between them.
- A collection of vectors is *orthonormal* if they are are pairwise orthogonal and have length 1. They are automatically linearly independent.
- If $\vec{a}_1, \ldots, \vec{a}_k$ in $\mathbb{R}^n$ are orthonormal, then the *orthogonal projection* onto their span is given by QQ^T, where Q is the matrix with columns given by the $\vec{a}_i$'s and Q^T is its *transpose*, obtained by switching the rows and columns (flipping the matrix).