MATH21B – LECTURE 11: LINEAR SPACES
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1. Linear spaces

Problem 1. Which of the following sets are linear spaces?

(a) The kernel $\ker(A)$ of a linear transformation $A\colon \mathbb{R}^m \to \mathbb{R}^n$.
(b) The set of invertible 2×2-matrices.
(c) All functions $f\colon \mathbb{R} \to \mathbb{R}$ in $C(\mathbb{R})$ such that $f(0) = 1$.
(d) All functions $f\colon \mathbb{R} \to \mathbb{R}$ in $C(\mathbb{R})$ such that $f(0) = f(1)$.
(e) All functions $f\colon \mathbb{R} \to \mathbb{R}$ in $C(\mathbb{R})$ such that $f(0) \neq f(1)$.
(f) The set of 3×3-matrices whose diagonal entries are 0.
(g) The set $C([0,1])$ of real-valued functions $f\colon [0,1] \to \mathbb{R}$, where $[0,1]$ denotes the set of $x \in \mathbb{R}$ such that $0 \leq x \leq 1$.

Solution. All are subsets of linear spaces, so we need to check that they contain $\vec{0}$, are closed under addition, and are closed under scaling. The sets (a), (d), (f) and (g) have these properties, so are linear spaces. The set (b) doesn't contain the 0 matrix (which has all entries 0), the sets (c) and (e) don't contain the 0 function (which taking value 0 for all x). ■

Problem 2. Give a basis for the following linear spaces:

(a) The polynomials p of degree ≤ 3 such that $p(0) = 0$.
(b) The (4×4)-matrices whose only non-zero entries are on the diagonal.
(c) The (2×2)-matrices whose diagonal sum is 0.

What is their dimension?

Solution. (a) The polynomials x, x^2 and x^3. It is 3-dimensional.
(b) The matrices with a single entry on the diagonal equal to 1 and 0's elsewhere. It is 4-dimensional.
(c) The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

It is 3-dimensional. ■

Problem 3 (Hard)**.** A real-valued function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *periodic* if $f(t+1) = f(t)$ for all $t \in \mathbb{R}$.

(i) Give an example of a periodic function.
(ii) Show that the set of real-valued periodic functions is a linear space.
(iii) Explain why we may think of these as real-valued periodic functions on the circle.

Solution. (i) $f(x) = \sin(2\pi x)$.
(ii) It is a linear subspace of $C(\mathbb{R})$, as one easily verifies the condition $f(t+1) = f(t)$ for all $t \in \mathbb{R}$ is satisfies by the function that is 0 everywhere, and preserved by scalar multiplication and addition.
(iii) If we parametrize the circle S^1 by angle $\theta \in [0, 2\pi)$, then there is a map $\pi\colon \mathbb{R} \to S^1$ given by $t \mapsto 2\pi t$ and function $f\colon S^1 \to \mathbb{R}$ gives a periodic function $f \circ \pi\colon \mathbb{R} \to \mathbb{R}$. Conversely, any periodic function f defines a function $\bar{f}\colon S^1 \to \mathbb{R}$ by sending θ to $f(2\pi\theta)$. ■

2. Linear transformations revisited

Problem 4. Let $P_n(\mathbb{R})$ be the linear space of polynomials with real coefficients of degree $\leq n$.

(i) Show that differentiation $D\colon P_n(\mathbb{R}) \to P_n(\mathbb{R})$ is a linear transformation.

(ii) What is the size of a matrix A for D?
(iii) What is the matrix A of D with respect to the basis $\{1, x, x^2, \ldots, x^n\}$?
(iv) What is the matrix B of $D - \mathrm{id}$ with respect to the $\{1, x, x^2, \ldots, x^n\}$?
(v) Is there any non-zero polynomial p which satisfies $\frac{d}{dx}p(x) = p(x)$?

Solution. (i) We shall verify this explicitly, instead of using the general properties of differentiation. We have that $D(0) = 0$. Writing $p(x) = \sum_i a_i x^i$ and $q(x) = \sum_i b_i x^i$, we have

$$D(p+q) = D\left(\sum a_i x^i + \sum b_i x^i\right) = \sum_i (a_i + b_i) i x^{i-1} = \sum_i a_i x^{i-1} + \sum_i b_i i x^{i-1} = D(p) + D(q).$$

Similarly, for $\lambda \in \mathbb{R}$ we have

$$D(\lambda p) = D\left(\sum \lambda a_i x^i\right) = \sum_i \lambda a_i i x^{i-1} = \lambda \sum_i a_i i x^{i-1} = \lambda D(p).$$

(ii) Since $P_n(\mathbb{R})$ is $(n+1)$-dimensional, A will be an $(n+1) \times (n+1)$-matrix.
(iii) The matrix A is obtained by evaluating D on the x^i and recording the coordinates of the result with respect to the basis as the columns. Thus the matrix is given by

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & n \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

(iv) The matrix is given by

$$\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 & n \\ 0 & 0 & \cdots & \cdots & 0 & -1 \end{bmatrix}$$

(v) This would be a non-zero element in $\ker(D - \mathrm{id})$. But the matrix is easily seen to have reduced row echelon form given by the identity matrix and thus have kernel $\{0\}$. ∎

Problem 5 (Hard)**.** Let $C^1(\mathbb{R})$ be the linear space of functions are at least once differentiable.

(i) Show that differentiation $D \colon C^1(\mathbb{R}) \to C(\mathbb{R})$ is a linear transformation.
(ii) Is there any non-zero function f which satisfies $\frac{d}{dx}f(x) = f(x)$?

Solution. (i) That $D(0) = 0$, $D(f+g) = D(f) + D(g)$ and $D(\lambda f) = \lambda D(f)$ are standard properties of differentiation.
(ii) Yes, $f(x) = c\exp(x)$ for any $c \neq 0$. ∎

Summary

- To give the structure of a *linear space* on a set V is to give ways to add elements and scale them, together with a zero vector $\vec{0}$ (this is somewhat imprecise, see Definition 4.1.1 of Bretscher for details).
- Most of our linear spaces V arise as linear subspaces and then we need to check that (i) $\vec{0}$ is in V, (ii) V is closed under addition, (iii) V is closed under scalar multiplication.
- The following are common examples of linear spaces:
  - The set $C(\mathbb{R})$ of continuous real-valued functions $f\colon \mathbb{R} \to \mathbb{R}$. The addition and scaling is pointwise, and $\vec{0}$ is the function that is 0 everywhere.
  - The set $P_n(\mathbb{R})$ of polynomials with real coefficients of degree $\leq n$. This is a linear subspace of $C(\mathbb{R})$.
  - The set $M_{m,n}(\mathbb{R})$ of $(m \times n)$-matrices with real entries. The addition and scaling are entry-wise, and $\vec{0}$ is the matrix with all entries 0.