MATH21B – LECTURE 10: COORDINATES
SPRING 2018, HARVARD UNIVERSITY

1. Putting a vector in other coordinates

Problem 1. Suppose we are given a new basis

$$\vec{a}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{a}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \vec{a}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix},$$

of $\mathbb{R}^3$. Let S be the (3×3)-matrix with columns $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

(i) Explain why S is invertible.
(ii) Compute S^{-1}.
(iii) Give the components of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ with respect to the new basis.
(iv) Give the components of

$$\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

with respect to the new basis. How can you verify your answer is correct?

Solution. (i) An $(n \times n)$-matrix is invertible if and only if it has rank n, which is equivalent to the columns being linearly independent. Here the columns are a basis, so by definition they are linearly independent

(ii) By computing $\mathrm{rref}([S \mid \mathrm{id}_3])$ we see that

$$S^{-1} = \begin{bmatrix}1/2 & 1/2 & -1/2\\-1/2 & 1/2 & 1/2\\1/2 & -1/2 & 1/2\end{bmatrix}.$$

(iii) One needs to compute $S^{-1}\vec{e}_1$, $S^{-1}\vec{e}_2$, and $S^{-1}\vec{e}_3$. These are just the columns of S^{-1}.

(iv) First $\vec{v}$ for this vector, then we need to compute $S^{-1}\vec{v}$. It is given by $(-2, 1, 1)^T$. To verify it, we need to check that $-2\vec{a}_1 + \vec{a}_2 + \vec{a}_3 = \vec{v}$, which is correct. ■

2. Putting a matrix in other coordinates

Problem 2. Again consider the basis $\vec{a}_1, \vec{a}_2, \vec{a}_3$ of Problem 1.

(i) Give the matrix B obtained by expressing the matrix A given below in terms of this new basis:

$$A = \begin{bmatrix}1 & 2 & 0\\0 & 1 & 2\\2 & 0 & 1\end{bmatrix}.$$

(ii) The answer to (i) should surprise you. Can you explain your answer?

Solution. (i) We have that $B = S^{-1}AS$ and thus we compute

$$B = \begin{bmatrix}1 & 2 & 0\\0 & 1 & 2\\2 & 0 & 1.\end{bmatrix}.$$

(ii) We have that B is the same as A! This happened accidentally, as $SA = AS$, so that $B = S^{-1}AS = S^{-1}SA = A$. This is of course very rare. ■

Problem 3. (i) Give the matrix B reflection in the yz-plane.

(ii) Give the matrix A for reflection in the plane orthogonal to the vector $(1,1,1)^T$. How can you verify your answer is correct?

Solution. (i) It is given by

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(ii) Let us take a basis given by $\vec{v}_1 = (1,1,1)^T$ and two vectors spanning the orthogonal complement to $(1,1,1)^T$, such as $\vec{v}_2 = (-1,1,0)^T$ and $\vec{v}_3 = (0,1,-1)^T$. Then geometrically, with respect to this basis the linear transformation is given by B, so that if S is matrix with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$, we have $A = SBS^{-1}$.

We compute that

$$S^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix}$$

and that

$$A = SBS^{-1} = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}.$$

We can verify e.g. by checking that $(1,1,1)^T$ gets sent to $(-1,-1,-1)^T$ by A. ■

Problem 4. Recall that for an $(n \times n)$-matrix A, the linear subspace $\mathrm{Fix}(A) \subset \mathbb{R}^4$ is given by the set $\{\vec{x} \mid A\vec{x} = \vec{x}\}$. Now consider the (4×4)-matrix

$$A = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(i) Show that $\mathrm{Fix}(A)$ is two-dimensional and give a basis.

(ii) What is the dimension of the orthogonal complement $\mathrm{Fix}(A)^\perp = \{\vec{x} \mid \vec{x} \cdot \vec{v} = \vec{0} \text{ for all } \vec{v} \in \mathrm{Fix}(A)\}$? Give a basis.

(iii) Show that your bases of $\mathrm{Fix}(A)$ and $\mathrm{Fix}(A)^\perp$ can be combined to a basis of $\mathbb{R}^4$.

(iv) Express A with respect to this basis.

(v) What is a geometric interpretation of A with respect to this basis?

Solution. (i) We need to compute

$$\mathrm{rref}(A - \mathrm{id}_4) = \begin{bmatrix} 1 & -1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and we read off that a basis is given by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(ii) By construction any vector in $\mathrm{Fix}(A)^\perp$ is linearly independent of all vectors in $\mathrm{Fix}(A)$, and any vector in $\mathbb{R}^4$ is a unique sum of a vector in $\mathrm{Fix}(A)$ and a vector in $\mathrm{Fix}(A)^\perp$. Thus the dimension must be 2.

For a basis it is equivalent to find a basis of the kernel of the matrix with rows the two vectors in (ii):

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ \sqrt{2} & 0 & 1 & 0 \end{bmatrix}$$

Its reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 1/\sqrt{2} & 0 \\ 0 & 1 & -1/\sqrt{2} & 0 \end{bmatrix}$$

so we get

$$\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(iii) Clear from (iii).

(iv) The matrix S will be

$$S = \begin{bmatrix} 1 & \sqrt{2} & -1/\sqrt{2} & 0 \\ 1 & 0 & 1/\sqrt{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and though its inverse is relatively complicated, one may compute that

$$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(v) It is a reflection in a 2-dimensional plane in 4-dimensional space. ■

Summary

- Switching to a different basis is useful when solving problems.
- When going from the standard $(\vec{e}_1, \ldots, \vec{e}_n)$ of $\mathbb{R}^n$ to $(\vec{a}_1, \ldots, \vec{a}_n)$ form the $(n \times n)$-matrix S with these vectors as columns. The components of a vector $\vec{v}$ with respect to the new basis are $S^{-1}\vec{v}$. A matrix A expressed in terms of the new basis is given by $B = S^{-1}AS$.
- If $B = S^{-1}AS$ then we say that B is *similar* to A.