MATH21B – LECTURE 1: INTRODUCTION AND LINEAR EQUATIONS
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1. Solving some equations

Problem 1. Given 3 bundles of superior paddy [unhusked rice], 2 bundles of ordinary paddy, and 1 bundle of inferior paddy, [together they yield] 39 dou of grain; 2 bundles of superior paddy, 3 bundles of ordinary paddy, and 1 bundle of inferior paddy [together yield] 34 dou of grain; 1 bundle of superior paddy, 2 bundles of ordinary paddy, and 3 bundles of inferior paddy [together yield] 26 dou of grain. Problem: 1 bundle of superior, ordinary, and inferior paddy each yield how much grain?

今有上禾三秉，中禾二秉，下禾一秉，實三十九斗；上禾二秉，中禾三秉，下禾一秉，實三十四斗；上禾一秉，中禾二秉，下禾三秉，實二十六斗。問上、中、下禾實一秉各幾何？

Solution. Let x denote the dou per bundle of superior paddy, y denote the dou per bundle of ordinary paddy, and z denote the dou per bundle of inferior paddy. The problems asks for a solution of the system of linear equations

$$\left| \begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26 \end{aligned} \right| .$$

In section we solved this using elimination, and found that there is a unique solution given by $x = \frac{37}{4}$, $y = \frac{17}{4}$, and $z = \frac{11}{4}$. ■

2. Systems of linear equations

Problem 2. (i) Which of the following are systems of linear equations?

$$\text{(a)} \left| \begin{aligned} 2x + 3y &= 5 \\ x + 4y &= 5 \end{aligned} \right| \qquad \text{(b)} \left| \begin{aligned} x_1^2 + 3x_2 - 3 &= 5 \\ x_1 + 4\sin(x_2) &= 5 \end{aligned} \right| \qquad \text{(c)} \left| 2x = 5 \right|$$

$$\text{(d)} \left| \begin{aligned} x_1 + 3x_2 - 3/x_2 &= 5 \\ 2x_1 + 4x_2 &= 5 \end{aligned} \right| \qquad \text{(e)} \left| \begin{aligned} x + y + z &= \cos(3) \\ x + y + z &= 0 \end{aligned} \right|$$

(ii) Solve the systems of linear equations among (a) – (e).

Solution. (i) Only (a), (c), and (e) are systems of linear equations. The system of equations (b) contains the terms x_1^2 and $\sin(x_2)$, which are not of the form (constant) $\cdot$ (unknown). Similarly, the system of equations (d) contains the term $-3/x_2$, which is not of the form (constant) $\cdot$ (unknown).

(ii) We shall always use elimination, starting with (a). In that case we first switch the equations

$$\left| \begin{aligned} 2x + 3y &= 5 \\ x + 4y &= 5 \end{aligned} \right| \xrightarrow{\text{I' = II, II' = I}} \left| \begin{aligned} x + 4y &= 5 \\ 2x + 3y &= 5 \end{aligned} \right| \xrightarrow{\text{II' = II-2·I, II' = I}} \left| \begin{aligned} x + 4y &= 5 \\ -5y &= -5 \end{aligned} \right|$$

and then subtract twice the first equation from the second. Next we multiple the second equation by $-1/5$ and add -4 times the second equation to the first:

$$\xrightarrow{\text{II' = }-1/5\text{II}} \left| \begin{aligned} x + 4y &= 5 \\ y &= 1 \end{aligned} \right| \xrightarrow{\text{I' = I}-4\text{II}} \left| \begin{aligned} x \qquad &= 1 \\ y &= 1 \end{aligned} \right| .$$

For (c), we only need to divide the first (and only) equation by 2 to get $\boxed{x = 5/2}$.

For (e), we subtract the first equation from the second:

$$\left|\begin{aligned} x+y+z &= \cos(3) \\ x+y+z &= 0 \end{aligned}\right| \xrightarrow{\text{II' = II-I}} \left|\begin{aligned} x+y+z &= \cos(3) \\ 0 &= -\cos(3) \end{aligned}\right|$$

and since $\cos(3) \neq 0$, we reach a contradiction: this system has no solutions. It is *inconsistent*. Geometrically, these are two distinct parallel planes which do not intersect. ■

Problem 3. (i) Give a system of two linear equations in two unknowns which has no solution.

(ii) Give a system of two linear equations in two unknowns which has infinitely many solutions.

(iii) Give a system of two linear equations in three unknowns which has infinitely many solutions.

Solution. (i) Take the system

$$\left|\begin{aligned} 2x+y &= 5 \\ 2x+y &= 0 \end{aligned}\right|$$

in the unknowns x and y. It has no solution since its solution set is the intersection of two distinct parallel lines, which is empty.

(ii) Take the system

$$\left|\begin{aligned} 2x+y &= 5 \\ 4x+2y &= 10 \end{aligned}\right|$$

in the unknowns x and y. It has infinitely many solutions since its solution set is the intersection of the same two lines, which is a line.

(iii) Take the system

$$\left|\begin{aligned} x \qquad &= 0 \\ y &= 0 \end{aligned}\right|$$

in the unknowns x, y and z. It has infinitely many solutions given by $(0, 0, z)$ with z arbitrary. Geometrically, its solution set the intersection of the planes $x = 0$ and $y = 0$, which is a line. ■