MATH21B – CIRCULANT MATRICES
SPRING 2018, HARVARD UNIVERSITY

1. Circulant matrix

Whenever A is circulant, you can write it as a linear combination of id, Q, Q^2, etc. As a concrete example let's take $A = 2 \cdot \mathrm{id} + 3 \cdot Q + 7 \cdot Q^3$ an (8×8)-matrix:

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 2 & 3 & 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 0 & 7 \\ 7 & 0 & 0 & 0 & 0 & 2 & 3 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 & 2 & 3 \\ 3 & 0 & 7 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Let's evaluate A on an eigenvector $\vec{v}$ of Q with eigenvalue $\lambda = \exp(k \cdot 2\pi i/n)$ (with $0 \leq k \leq n-1$). Such a $\vec{v}$ is given as follows, but we will not need this:

$$\begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \end{bmatrix}.$$

You will realize that $\vec{v}$ is an eigenvector of A but for a different eigenvalue. In our example:

$$A\vec{v} = (2 \cdot \mathrm{id} + 3 \cdot Q + 7 \cdot Q^3)\vec{v} = 2 \cdot \vec{v} + 3\lambda \cdot \vec{v}v + 7\lambda^3 \cdot \vec{v} = (2 + 3\lambda + 7\lambda^3)\vec{v}$$

says that $\vec{v}$ is eigenvector but for eigenvalue $2 + 3\lambda + 7\lambda^3$.

Thus the conclusion is:

- the eigenvectors of a circulant matrix A are those of Q,
- the eigenvalues of $A = c_0 \cdot \mathrm{id} + c_1 \cdot Q + c_2 \cdot Q^2 + c_3 \cdot Q^3 + \cdots$ are given by $c_0 + c_1 \cdot \lambda + c_2 \cdot \lambda^2 + c_3 \cdot \lambda^3 + \cdots$ with λ the eigenvalues of Q.