

THREE LECTURES ON TOPOLOGICAL MANIFOLDS

ALEXANDER KUPERS

ABSTRACT. We introduce the theory of topological manifolds (of high dimension). We develop two aspects of this theory in detail: microbundle transversality and the Pontryagin-Thom theorem.

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These are the notes for three lectures I gave at a workshop. They are mostly based on Kirby-Siebenmann [KS77] (still the only reference for many basic results on topological manifolds), though we have eschewed PL manifolds in favor of smooth manifolds and often do not give results in their full generality.

1. LECTURE 1: THE THEORY OF TOPOLOGICAL MANIFOLDS

Definition 1.1. A *topological manifold of dimension n* is a second-countable Hausdorff space M that is locally homeomorphic to an open subset of \mathbb{R}^n .

In this first lecture, we will discuss what the “theory of topological manifolds” entails. With “theory” I mean a collection of definitions, tools and results that provide insight in a mathematical structure. For smooth manifolds of dimension ≥ 5 , handle theory provides such insight. We will describe the main tools and results of this theory, and then explain how a similar theory may be obtained for topological manifolds. In an intermezzo we will give the proof of the Kister’s theorem, to give a taste of the infinitary techniques particular to topological manifolds. In the second lecture we will develop one of the important tools of topological manifold theory — transversality — and in the third lecture we will obtain one of its important results — the classification of topological manifolds of dimension ≥ 6 up to bordism.

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1.1. The theory of smooth manifolds. There exists a well-developed theory of smooth manifolds, see [Kos93, Hir94, Wal16]. Though we will recall the basic definition of a smooth manifold to refresh the reader’s memory, we will not recall most other definitions, e.g. those of smooth manifolds with boundary or smooth submanifolds.

Definition 1.2. A *smooth manifold of dimension n* is a topological manifold of dimension n with the additional data of a smooth atlas: this is a maximal compatible collection of map $\phi_i: \mathbb{R}^n \supset U_i \hookrightarrow M$ that are homeomorphisms onto their image, called *charts*, such that the *transition functions* $\phi_j^{-1}\phi_i$ between these charts are smooth.

The goal of the theory of smooth manifolds is to classify smooth manifolds, as well as various geometric objects in and over them, such as submanifolds, vector bundles or special metrics. This was to a large extent achieved by the high-dimensional theory developed in the 50’s, 60’s and 70’s. Here “high dimension” means dimensions $n \geq 5$, though we will assume $n \geq 6$ for simplicity. The insight of this theory is to smooth manifolds can be understood by building them out of standard pieces called *handles*, see e.g. [LÖ2].

Definition 1.3. Let M be a smooth manifold with boundary ∂M . Given a smooth embedding $\phi: S^{i-1} \times D^{n-i} \hookrightarrow \partial M$, we define a new manifold

$$M' := M \cup_{\phi} D^i \times D^{n-i}$$

Then M' is said to be *the result of attaching an i -handle to M* .

Giving a handle decomposition of M means expressing M as being given by iterated handle attachments, starting at \emptyset .

Remark 1.4. When we glue $D^i \times D^{n-i}$ to M corners may appear. These can canonically be smoothed, though we shall ignore such difficulties as they play no role in the topological case.

Example 1.5. A sphere S^n has a handle decomposition with only two handles: a 0-handle $D^0 \times D^n \cong D^n$ (attached along $S^{-1} \times D^n = \emptyset$ to the empty manifold). We then attach an n -handle $D^n \cong D^n \times D^0$, by gluing its boundary S^{n-1} to the boundary S^{n-1} of the 0-handle via the identity map.

It is very important to specify the gluing map. In high dimensions there exist so-called exotic spheres, which are homotopy equivalent to S^n but not diffeomorphic to it, and these are obtained by gluing an n -handle to a 0-handle along exotic diffeomorphisms $S^{n-1} \rightarrow S^{n-1}$. For example, for $n = 7$, there are 28 exotic spheres.

Exercise 1.1. Give a handle decomposition of the genus g surface Σ_g with $2g + 2$ handles and show there is no handle decomposition with fewer handles.

The following existence result is often proven using Morse theory, i.e. the theory of generic smooth functions $M \rightarrow \mathbb{R}$ and their singularities, see e.g. [Mil65].

Theorem 1.6. *Every smooth manifold admits a handle decomposition.*

Once we have established the existence of handle decompositions, we should continue the developed of the high-dimensional theory by understanding the manipulation of handle decompositions. The goal is to reduce questions about manifolds to questions about homotopy

theory or algebraic K-theory, both of which are more computable than geometry. The highlights among the results are as follows, stated vaguely:

- the *Pontryagin-Thom theorem*: the homeomorphism classes of smooth manifolds up to an equivalence relation called *bordism* (which is closely related to handle attachments) can be computed homotopy-theoretically [Hir94, Chapter 7].
- the *h-cobordism theorem*: if a manifold M of dimension ≥ 6 looks like a product $N \times I$ from the point of view of homotopy theory and algebraic K-theory, it is diffeomorphic to $N \times I$ [Mil65].
- the *end theorem*: if an open manifold M of dimension ≥ 5 looks like the interior of a manifold with boundary from the point of view of homotopy theory and algebraic K-theory, then it is the interior of a manifold with boundary [Sie65].
- *surgery theory*: a smooth manifold of dimension ≥ 5 is described by a space with Poincaré duality, bundle data and simple-homotopy theoretic data, satisfying certain conditions [Wal99]. In surgery theory, one link between homotopy theory and manifolds is through the identification of *normal invariants* using the Pontryagin-Thom theorem mentioned above.

To start manipulating handles, we use *transversality*. A handle $D^i \times D^{n-i}$ has a *core* $D^i \times \{0\}$ and *cocore* $\{0\} \times D^{n-i}$, which form the essential parts of the handle. We need to control the way that the boundary of a core of one handle intersects the cocore of other handles. Transversality results allow us assume these intersections are transverse, so we can start manipulating them. This is one of the two reasons we discuss transversality in these lectures, the other being the important role it plays in the proof of the Pontryagin-Thom theorem in the next section.

Remark 1.7. In addition to smooth manifolds, there is a second type of manifold that has a well-developed theory: *piecewise linear manifolds*, usually shortened to PL-manifolds [RS72]. This is a topological manifold with a maximal piecewise linear atlas. Here instead of the transition functions $\phi_j^{-1}\phi_i$ being smooth, they are required to be piecewise linear.

The results of this theory are essentially the same as for those in the theory of smooth manifolds, though they differ on a few points in their statements and the methods used to prove them. We will rarely mention PL-manifolds in these notes, but in reality they play a more important role in the theory of topological manifolds than smooth manifolds. This is because topological manifolds are closer to PL manifolds than smooth manifolds.

1.2. The theory of topological manifolds. The theory of topological manifolds is modeled on that of smooth manifolds, using the existence and manipulation of handles. Hence the final definitions, tools and theorems that we want for topological manifolds are similar to those of smooth manifolds.

To obtain this theory, we intend to bootstrap from smooth or PL manifolds, a feat that was first achieved by Kirby and Siebenmann [KS77, Essay IV]. They did this by understanding smooth and PL structures on (open subsets of) topological manifolds. To state the results of Kirby and Siebenmann, we define three equivalence relations on smooth structures on (open subsets of) a topological manifold M , which one should think of as an open subset of a larger topological manifold.

To give these equivalence relations, we have to explain how to pull back smooth structures along topological embeddings of codimension 0. In this case a topological embedding is just a continuous map that is a homeomorphism onto its image, and a typical example is the inclusion of an open subset. If Σ is a smooth structure on M and $\varphi: N \rightarrow M$ is a codimension 0 topological embedding, then $\varphi^*\Sigma$ is the smooth structure given by the maximal atlas containing the maps $\varphi^{-1} \circ \phi_i: \mathbb{R}^n \subset U_I \hookrightarrow \varphi(N) \rightarrow N$ for those charts ϕ_i of Σ with $\phi_i(U_i) \subset \varphi(N) \subset M$. Note that if N and M were smooth, φ is smooth if and only if $\varphi^*\Sigma_M = \Sigma_N$.

Definition 1.8. Let M be a topological manifold.

- Two smooth structures Σ_0 and Σ_1 are said to be *concordant* if there is a smooth structure Σ on $M \times I$ that near $M \times \{i\}$ is a product $\Sigma_i \times \mathbb{R}$.
- Σ_0 and Σ_1 are said to be *isotopic* if there is a (continuous) family of homeomorphisms $\phi_t: [0, 1] \rightarrow \text{Homeo}(M)$ such that $\phi_t = \text{id}$ and $\phi_1^*\Sigma_0 = \Sigma_1$ (i.e. the atlases for Σ_0 and Σ_1 are compatible).
- Σ_0 and Σ_1 are said to be *diffeomorphic* if there is a homeomorphism $\phi: M \rightarrow M$ such that $\phi^*\Sigma_0 = \Sigma_1$.

Remark 1.9. By the existence of smooth collars, Σ_0 and Σ_1 are concordant if and only if there is a smooth structure Σ on $M \times I$ so that $\Sigma|_{M \times \{i\}} = \Sigma_i$, where implicitly we are saying that the boundary of M is smooth.

Note that isotopy implies concordance and isotopy implies diffeomorphism. Kirby and Siebenmann proved the following foundational results about smooth structures:

- *concordance implies isotopy*: If $\dim M \geq 6$, then the map

$$\frac{\{\text{smooth structures on } M\}}{\text{isotopy}} \longrightarrow \frac{\{\text{smooth structures on } M\}}{\text{concordance}}$$

is a bijection. Hence also concordance implies diffeomorphism:

$$\begin{array}{ccc} & \text{isotopy} & \\ \swarrow & & \searrow \\ \text{diffeomorphism} & & \text{concordance} \end{array}$$

- *concordance extension*: Let M be a topological manifold of dimension ≥ 6 with a smooth structure Σ_0 and $U \subset M$ open. Then any concordance of smooth structures on U starting at $\Sigma_0|_U$ can be extended to a concordance of smooth structures on M starting at Σ_0 .
- the *product structure theorem*: If $\dim M \geq 5$, then taking the cartesian product with \mathbb{R} induces a bijection

$$\frac{\{\text{smooth structures on } M\}}{\text{concordance}} \xrightarrow{- \times \mathbb{R}} \frac{\{\text{smooth structures on } M \times \mathbb{R}\}}{\text{concordance}}.$$

These theorems can be used to prove a classification theorem for smooth structures.

Theorem 1.10 (Smoothing theory). *There is a bijection*

$$\frac{\{\text{smooth structures on } M\}}{\text{concordance}} \xrightarrow{\tau} \frac{\{\text{lifts of } TM: M \rightarrow B\text{Top to } BO\}}{\text{vertical homotopy}}.$$

These techniques are used as follows: every point in a topological manifold has a neighborhood with a smooth structure. This means we can use all our smooth techniques locally. Difficulties arise when we want to move to the next chart. Using the above three theorems, it is sometimes possible to adapt the smooth structures so that we can transfer certain properties. We will do this for handlebody structures in this lecture and microbundle transversality in the next lecture. Perhaps the slogan to remember is a sentence in Kirby-Siebenmann (though we will replace “PL” by “smoothly”):

The intuitive idea is that ... the charts of a TOP manifolds ... are *as good as PL compatible*.

Let me try to somewhat elucidate the hardest part of proving these results: it appears in the product structure theorem (which also happens to be the theorem we will use most in later lectures). The proof of this theorem is by induction over charts, and the hardest step is the initial case $M = \mathbb{R}^n$. We prove this by showing that for $n \geq 6$, both \mathbb{R}^n and \mathbb{R}^{n+1} have a unique smooth structure up to concordance; a map between two sets containing a single element is of course a bijection. To produce a concordance from any smooth structure of \mathbb{R}^n to the standard one, we will need both the *stable homeomorphism theorem* and *Kister’s theorem*. Let us state these results.

Definition 1.11. A homeomorphism $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *stable* if it is a finite composition of homeomorphisms that are identity on some open subset of \mathbb{R}^n .

Remark 1.12. Using the topological version of isotopy extension [EK71], one may prove that f is stable if and only if for all $x \in \mathbb{R}^n$ there is an open neighborhood U of x such that $f|_U$ is isotopic to a linear isomorphism. This allows one to define a notion of a stable atlas and stable manifold, which play a role in the proof of stable homeomorphism theorem.

The following is due to Kirby [Kir69].

Theorem 1.13 (Stable homeomorphism theorem). *If $n \geq 6$, then every orientation-preserving homeomorphism of \mathbb{R}^n is stable.*

Remark 1.14. The case $n = 0, 1$ are folklore, $n = 2$ follows from work by Radó [Rad24], $n = 3$ from work by Moise [Moi52], and the cases $n = 4, 5$ are due to Quinn [Qui82].

Exercise 1.2. Prove the case $n = 1$ of the stable homeomorphism theorem.

Lemma 1.15. *A stable homeomorphism is isotopic to the identity.*

Proof. If a homeomorphism h is the identity near $p \in \mathbb{R}^n$, and $\tau_p: \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the translation homeomorphism $x \mapsto x + p$, then $\tau_p^{-1}h\tau_p$ is the identity near 0. Thus the formula

$$[0, 1] \ni t \mapsto \tau_{t \cdot p}^{-1} h \tau_{t \cdot p}$$

gives an isotopy from h to a homeomorphism that is the identity near 0.

Next suppose we are given a stable homeomorphism h , which by definition we may write as $h = h_1 \cdots h_k$ with each h_i a homeomorphism that is the identity on some open subset U_i . Then applying the above construction to each of the h_i using $p_i \in U_i$, shows that h is isotopic to a homeomorphism h' that is the identity near 0.

Finally, let $\sigma_r: \mathbb{R}^n \rightarrow \mathbb{R}^n$ for $r > 0$ denote the scaling homeomorphism given by $x \mapsto rx$ then

$$[0, 1] \ni r \mapsto \begin{cases} \sigma_{1-r}^{-1} h' \sigma_{1-r} & \text{if } r \in [0, 1) \\ \text{id} & \text{if } r = 1 \end{cases}$$

gives an isotopy from h' to the identity (note that for continuity in the compact open topology we require convergence on compacts). \square

Thus the stable homeomorphism theorem implies that $\text{Homeo}(\mathbb{R}^n)$ has two path components: one contains the identity, the other the orientation-reversing map $(x_1, x_2, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$. We will combine this with Kister's theorem [Kis64]. This result uses the definition of a topological embedding, which in this case — when the dimensions are equal — is just a continuous map that is a homeomorphism onto its image.

Theorem 1.16 (Kister's theorem). *Every topological embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^n$ is isotopic to a homeomorphism. In fact, the proof gives a canonical such isotopy, which depends continuously on the embedding.*

In the intermezzo following this section, we shall give a proof of this theorem. It is proven by what may be described impressionistically as a “convergent infinite sphere jiggling” procedure. Combining this with Theorem 1.13 and Lemma 1.15 we obtain:

Corollary 1.17. *If $n \geq 6$, every orientation-preserving topological embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^n$ is isotopic to the identity.*

Corollary 1.18. *For $n \geq 6$, every smooth structure Σ on \mathbb{R}^n is concordant to the standard one.*

Proof. Any smooth chart for the smooth structure Σ can be used to obtain a smooth embedding $\phi_0: \mathbb{R}_{\text{std}}^n \hookrightarrow \mathbb{R}_\Sigma^n$, which without loss of generality we may assume to be orientation-preserving (otherwise precompose it with the map $(x_1, x_2, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$). A smooth embedding is in particular a topological embedding and by Corollary 1.17, it is isotopic to the identity. Pulling back the smooth structure along this isotopy gives us a concordance of smooth structures starting at the standard one and ending at Σ . \square

Remark 1.19. In particular, for $n \geq 6$ there is a unique smooth structure on \mathbb{R}^n up to diffeomorphism, as concordance implies isotopy implies diffeomorphism. The smooth structure on \mathbb{R}^n is in fact unique in all dimension except 4. The stable homeomorphism theorem and Kister's theorem are true even in dimension 4, but concordance implies isotopy fails.

Remark 1.20. The proof of the stable homeomorphism theorem is beautiful, but has many prerequisites. It relies on both the smooth end theorem and the classification of PL homotopy tori using PL surgery theory [KS77, Appendix V.B] [Wal99, Chapter 15A], and uses a so-called torus trick to construct a compactly-supported homeomorphism of \mathbb{R}^n agreeing with the original one on an open subset.

1.3. Existence of handle decompositions. As an application of the product structure theorem, we will prove that every topological manifold of dimension ≥ 6 admits a handle decomposition [KS77, Theorem III.2.1]. The definition of handle attachments and handle

decompositions for topological manifolds are as in Definition 1.3 except ϕ now only needs to be a topological embedding.

This definition involves topological manifolds with boundary, which are locally homeomorphic to $[0, \infty) \times \mathbb{R}^{n-1}$ and the points which correspond to $\{0\} \times \mathbb{R}^{n-1}$ form the boundary ∂M of M . We prove in Lemma 1.23 that every topological manifold with boundary admits a *collar*, i.e. a map $\partial M \times [0, \infty) \hookrightarrow M$ that is the identity on the boundary and a homeomorphism onto its image. Suppose one has a map $e: M' \hookrightarrow M$ of a topological manifold with boundary M' into a topological manifold M of the same dimension that is a homeomorphism onto its image. Using a collar for M' , we may isotope the map e such that its boundary $\partial M'$ has a *bicollar* in M , i.e. there is a map $\partial M \times \mathbb{R} \hookrightarrow M$ that is the identity on $\partial M \times \{0\}$ and is a homeomorphism onto its image. This isotopy is given by “pulling M back into its collar a bit.”

Remark 1.21. Not every map $e: M' \hookrightarrow M$ as above admits a bicollar, e.g. the inclusion of the closure of one of the components of the complement of the Alexander horned sphere, see Remark 3.10.

Theorem 1.22. *Every topological manifold M of dimension $n \geq 6$ admits a handle decomposition.*

Proof. Let us prove this in the case that M is compact. Then there exists a finite cover of M by closed subsets A_i , each of which is contained in an open subset U_i that can be given a smooth structure Σ_i (these do not have to be compatible). For example, one may obtain this by taking a finite subcover of the closed unit balls in charts.

By induction over i we construct a handlebody $M_i \subset M$ whose interior contains $\bigcup_{j \leq i} A_j$, starting with $M_{-1} = \emptyset$. So let $i \geq 0$ and suppose we have constructed M_{i-1} , then we will construct M_i . By the remarks preceding this theorem we assume that there exists a bicollar C_i of $\partial(M_{i-1} \cap U_i)$ in U_i . In particular C_i is homeomorphic to $\partial(M_{i-1} \cap U_i) \times \mathbb{R}$ and being an open subset of U_i with smooth structure Σ_i , admits a smooth structure. Thus we may apply the product structure theorem to $\partial(M_{i-1} \cap U_i) \subset C_i$, and modify the smooth structure on C_i by a concordance so that $\partial(M_{i-1} \cap U_i)$ becomes a smooth submanifold. By concordance extension we may then extend the concordance and resulting smooth structure to U_i . We can then use the relative version of the existence of handle decompositions for smooth manifolds to find a $N_i \subset U_i$ obtained by attaching handles to $M_{i-1} \cap U_i$, which contains a neighborhood of $A_i \cap U_i$. Taking $M_i := M_{i-1} \cup N_i$ completes the induction step. \square

Exercise 1.3. Modify the proof of Theorem 1.22 to work for non-compact M (hint: use that M is paracompact).

We now prove the lemma about collars. Though this is longer than the above proof, it is completely elementary.

Lemma 1.23. *The boundary of a topological manifold admits a collar $c: \partial M \times [0, 1) \hookrightarrow M$, unique up to isotopy.*

Proof. A local version of uniqueness can be used to prove existence. Its statement involves local collars. A *local collar* is an open subset $U \subset \partial M$ and a map $c: U \times [0, \infty) \rightarrow M$ that is the identity on $U \times \{0\}$ and a homeomorphism onto its image.

Then the local version of uniqueness says the following: given two local collars $c, d: U \times [0, \infty) \rightarrow M$ with the same domain, a closed $D_{\text{todo}} \subset U$ and open neighborhood $V_{\text{todo}} \subset M$ containing $D_{\text{todo}} \times \{0\}$, there exists a isotopy h_s of M , i.e. a family of homeomorphisms indexed by $s \in [0, 1]$, with the following properties:

- (i) $h_0 = \text{id}_M$,
- (ii) h_s is the identity on ∂M for all $s \in [0, 1]$,
- (iii) $h_1 d = c$ near $D_{\text{todo}} \times \{0\}$ in $U \times [0, \infty)$,
- (iv) h_s is supported in V_{todo} for all $s \in [0, 1]$.

To prove this local uniqueness statement, we do a ‘‘sliding procedure.’’ Let $W := d^{-1}(c(U \times [0, \infty)))$, an open subset of $U \times [0, \infty)$ containing $D \times \{0\}$. There exists an open subset of the form $W' \times [0, \epsilon) \subset W$ in $U \times [0, \infty)$ containing $D \times \{0\}$. We can now look at $d|_{W' \times [0, \epsilon)}$ in c -coordinates, i.e. consider it as a map $W' \times [0, \epsilon) \rightarrow U \times [0, \infty)$. It is now helpful to enlarge the manifolds we work in: $U \times [0, \infty) \subset U \times \mathbb{R}$ and $W' \times [0, \epsilon) \subset W' \times (-\epsilon, \epsilon)$, extending d by the identity to $\tilde{d}: W' \times (-\epsilon, \epsilon) \rightarrow U \times \mathbb{R}$. There exists a 1-parameter family of homeomorphisms σ_s of $U \times \mathbb{R}$ indexed by $s \in [0, 1]$, which ‘‘slides’’ along the lines $\{u\} \times \mathbb{R}$, satisfying

- (i) $\sigma_0 = \text{id}_{U \times \mathbb{R}}$,
- (ii) $\sigma_s(u, t) \in u \times \mathbb{R}$ and $\sigma_s(u, t) \in u \times [0, \infty)$ if $t \in [0, \infty)$,
- (iii) $\sigma_1(D \times \{0\}) \subset U \times (0, \infty)$,
- (iv) σ_t is supported on a closed neighborhood of $D \times \{0\}$ contained in $d^{-1}(V_{\text{todo}})$.

Hint for the construction of σ_s : find a continuous function $U \rightarrow [0, \infty)$ that is non-zero on D and whose graph lies in $d^{-1}(V_{\text{todo}}) \cup (U \times \{0\})$.

Then consider the family

$$\tilde{d}_s := s \mapsto \sigma_s \circ \tilde{d} \circ \sigma_s^{-1}$$

so that $\tilde{d}_0 = \tilde{d}$ and \tilde{d}_1 is the identity near $D \times \{0\}$, i.e. coincides with c (since we were working in c -coordinates). Now we can define the desired isotopy h_s by $\tilde{d}_s \circ d^{-1}$ on $d(W' \times [0, \epsilon))$ and identity elsewhere. This is continuous by properties (ii) and (iv) of σ_s . Properties (i), (iii) and (iv) of h_s follows from properties (i), (iii) and (iv) of σ_s respectively. Property (ii) of h_s follows from the definitions.

One deduces existence from uniqueness as follows. Since ∂M is paracompact, there exists a locally finite collection of closed sets A_β inside open subsets $U_\beta \subset \partial M$ so that $\bigcup_\beta A_\beta = \partial M$ and for all U_β we have a local collar $c_i: U_i \times [0, \infty) \hookrightarrow M$. Without loss of generality, the images of c_i are also locally finite, by shrinking them. Enumerate the β and write them as $i \in \mathbb{N}$ from now on. Our construction of the collar will by induction.

That is, suppose we have constructed $d_j: V_j \times [0, \infty) \hookrightarrow M$ on a neighborhood V_j of $\bigcup_{i \leq j} A_i$. Then for the subset $\partial M_j := V_j \cap U_{j+1}$ of the boundary we have two different local collars, c_{j+1} and d_j . We apply the local uniqueness with $c = c_{j+1}|_{\partial M_j \times [0, \infty)}$ and $d = d_j|_{\partial M_j \times [0, \infty)}$, $D_{\text{todo}} = A_{j+1} \cap \partial M_j$ and $V_{\text{todo}} = c_{j+1}(U_{j+1} \times [0, \infty))$. Then we get our isotopy h_t and a neighborhood $W_{j+1} \times [0, \epsilon) \subset \partial M_j \times [0, \infty)$ of $A_{j+1} \cap \partial M_j$ on which $h_1 d_j$ equals c_{j+1} . There is an open subset of $W'_{j+1} \times [0, \epsilon)$ of U_{j+1} containing A_{j+1} so that

$W'_{j+1} \cap \partial M_j \subset W_{j+1}$. Then $V_{j+1} := V_j \cup W'_{j+1}$ and define a function d'_{j+1} by

$$d'_{j+1}(m, t) := \begin{cases} c_{j+1}(m, t) & \text{if } m \in W'_{j+1} \\ h_1 d_j(m, t) & \text{otherwise} \end{cases}$$

This is a collar near $V_{j+1} \times \{0\}$, not necessarily everywhere. This may be fixed by shrinking it in V_{todo} , and thus we have obtained our next local collar $d_{j+1}: V_{j+1} \times [0, \infty) \hookrightarrow M$, completing the induction step. The local finiteness conditions and our control on the support means that the collar is modified near a fixed $m \in \partial M$ only a finite number of times. \square

1.4. Remarks on low dimensions. What happens in dimensions $n \leq 4$? The situation is different for $n \leq 3$ and $n = 4$. In dimensions 0, 1, 2 and 3, results of Radó and Moise say that every topological manifold admits a smooth structure unique up to isotopy. Informally stated, topological manifolds are the same as smooth manifolds.

Dimension 4 is more complicated [FQ90]. A result of Freedman uses an infinite collapsing construction to show that an important tool of high dimensions, the *Whitney trick*, still works for topological 4-manifolds with relatively simple fundamental group [Fre82]. This means that the higher-dimensional theory to a large extent applies to topological 4-manifolds. One important exception is that they may no longer admit a handle decomposition, though in practice one can work around this using the fact that a path-connected topological 4-manifold is smoothable in the complement of a point, see Section 8.2 of [FQ90].

In dimension 4, smooth manifolds behave differently. For example, $n = 4$ is the only case in which \mathbb{R}^n admits more than one smooth structure up to diffeomorphism. In fact, it admits uncountably many. Furthermore, these can occur in families: in all dimensions $\neq 4$ a submersion that is topologically a fiber bundle is smoothly a fiber bundle by the Kirby-Siebenmann bundle theorem [KS77, Essay II], but in dimension 4 there exists a smooth submersion $E \rightarrow [0, 1]$ with E homeomorphic to $\mathbb{R}^4 \times [0, 1]$ such that all fibers are non-diffeomorphic smooth structures on \mathbb{R}^4 [DMF92]. See also [Tau87]. Hence the product structure theorem is false for 3-manifolds, as it also involves 4-manifolds. In particular, it predicts two smooth structures on S^3 , but the now proven Poincaré conjecture says there is only one. However, it may not be the right intuition to think that there are different exotic smooth structures on a given manifold, but that many different smooth manifolds happen to be homeomorphic.

2. INTERMEZZO: KISTER'S THEOREM

In this section we give a proof of Kister's theorem on self-embeddings of \mathbb{R}^n , which we use in two places: (i) to prove the initial case of the product structure theorem, and (ii) to prove Theorem 3.7 about the existence of \mathbb{R}^n -bundles in microbundles. The proof is rather elementary, and a close analogue plays an important role in the ϵ -Schoenflies theorem used in [EK71] to prove isotopy extension for locally flat submanifolds.

The goal is to compare the following two basic objects of manifold theory:

- (i) The topological groups

$$\{\text{CAT-isomorphisms } (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)\}$$

with $\text{CAT} = \text{Diff}$ or Top . In words, these are the diffeomorphisms, homeomorphisms and PL-homeomorphisms of \mathbb{R}^n fixing the origin. These are denoted $\text{Diff}(n)$ and $\text{Top}(n)$ respectively.

(ii) The topological monoids

$$\{\text{CAT-embeddings } (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)\}$$

with $\text{CAT} = \text{Diff}$ or Top . In words, these are the self-embeddings of \mathbb{R}^n fixing the origin, either smooth or topological. There is no special notation for them, so we use $\text{Emb}_0^{\text{CAT}}(\mathbb{R}^n, \mathbb{R}^n)$.

Remark 2.1. We can include these spaces into the topological groups or monoids of CAT-isomorphisms or CAT-embeddings that do not necessarily fix the origin. A translation homotopy, i.e. deforming the embedding or isomorphism ϕ through the family

$$[0, 1] \ni t \mapsto \phi_t(x) := \phi(x) - t \cdot \phi(0)$$

shows that these inclusions are homotopy equivalences.

The reason there is no special notation for the self-embeddings is the following theorem.

Theorem 2.2. *The inclusion*

$$\{\text{CAT-isomorphisms } (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)\} \hookrightarrow \{\text{CAT-embeddings } (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)\}$$

is a weak equivalence if CAT is Diff or Top.

In this section we prove this theorem, and the smooth case will serve as an explanation of the proof strategy for the topological case.

Remark 2.3. The PL version of Theorem 2.2 is also true [KL66].

2.1. Smooth self-embeddings of \mathbb{R}^n . We start by proving that all of the inclusions

$$O(n) \hookrightarrow \text{GL}_n(\mathbb{R}) \hookrightarrow \text{Diff}(n) \hookrightarrow \text{Emb}_0^{\text{Diff}}(\mathbb{R}^n, \mathbb{R}^n)$$

are weak equivalences. Our strategy will be to prove that $\text{GL}_n(\mathbb{R}) \hookrightarrow \text{Emb}_0^{\text{Diff}}(\mathbb{R}^n, \mathbb{R}^n)$ is a weak equivalence, and note that our proof will restrict to a proof for $\text{GL}_n(\mathbb{R}) \hookrightarrow \text{Diff}(n)$. This will follow from the fact that a smooth map is a diffeomorphism if and only if it is surjective smooth embedding, which follows from the inverse function theorem. That $O(n) \hookrightarrow \text{GL}_n(\mathbb{R})$ is a weak equivalence is well-known and a consequence of Gram-Schmidt orthonormalization. By the 2-out-of-3 property for weak equivalences, this implies all inclusions are weak equivalences.

Theorem 2.4. *The inclusion*

$$\text{GL}_n(\mathbb{R}) \hookrightarrow \text{Emb}_0^{\text{Diff}}(\mathbb{R}^n, \mathbb{R}^n)$$

is a weak equivalence.

Proof. Recall that a weak equivalence is a map that induces an isomorphism on all homotopy groups with all base points, which is equivalent to the existence of dotted lifts for each

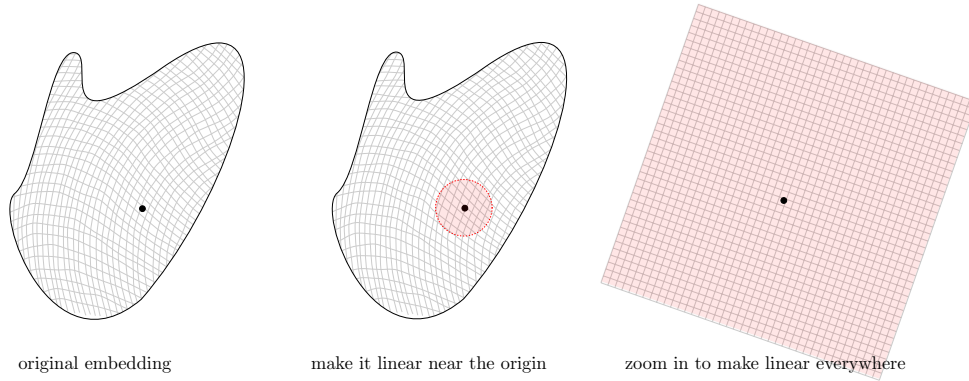


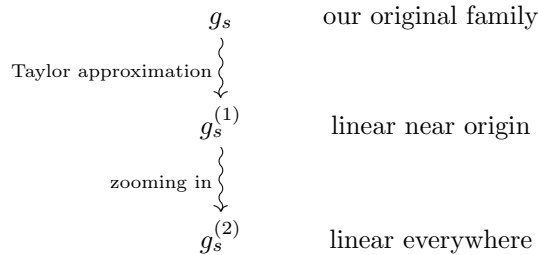
FIGURE 1. An outline for the strategy for Theorem 2.4.

$k \geq -1$ ($S^{-1} = \emptyset$ by convention) in each diagram

$$\begin{array}{ccc}
 S^k & \longrightarrow & \text{GL}_n(\mathbb{R}) \\
 \downarrow & \nearrow g_s^{(2)} & \downarrow \\
 D^{k+1} & \xrightarrow{g_s} & \text{Emb}_0^{\text{Diff}}(\mathbb{R}^n, \mathbb{R}^n)
 \end{array}$$

after homotoping the diagram. That is, we want to deform the family $(g_s)_{s \in D^{k+1}}$ into linear maps, staying in linear maps if we are already in linear maps. In this case, we will in fact not modify any embedding that is already linear.

Our strategy is as follows, see also Figure 1:



- (i) Our first step makes g_s linear near the origin. Let $\eta: [0, \infty) \rightarrow [0, 1]$ be a decreasing C^∞ function that is 1 near 0 and 0 on $[1, \infty)$ (it looks like a bump near the origin). Consider the function

$$G_s^{(1)}(x, t) := (1 - t\eta(\|x\|/\epsilon))g_s(x) + t\eta(\|x\|/\epsilon)Dg_s(0) \cdot x$$

with $s \in D^{k+1}$, $x \in \mathbb{R}^n$ and $t \in [0, 1]$. This is a candidate for the isotopy deforming g_s to be linear near the origin, but we still need to pick ϵ . We have that $G_s(x, t) = g_s(x)$ for all $t \in [0, 1]$ and $\|x\| \geq \epsilon$. For $\|x\| \leq \epsilon$ the difference $\|DG_s^{(1)}(x, t) - Dg_s(x)\|$ can be estimated as $\leq D\epsilon$ for D uniform in s . So picking ϵ small enough we can guarantee that $DG_s(x, t) \neq 0$ for all $x \in \mathbb{R}^n$, $t \in [0, 1]$. This implies that they are all embeddings,

since they are for $t = 0$. Indeed, the number of points in an inverse image cannot change without a critical point appearing. We set

$$g_s^{(1)} := G_s^{(1)}(-, 1)$$

and note that if g_s was linear, then $g_s = G_s^{(1)}(-, t)$ for all t (so in particular $g_s^{(1)}$ is linear).

(ii) Our second step zooms in on the origin. To do this, we define

$$G_s^{(2)}(x, t) := \begin{cases} Dg_s^{(1)}(0) \cdot x & \text{if } \|x\| \leq \epsilon \text{ and } t < 1, \text{ or if } t = 1 \\ (1-t)g_s^{(1)}\left(\frac{x}{1-t}\right) & \text{if } \|x\| > \epsilon \text{ and } t < 1 \end{cases}$$

We set $g_s^{(2)} := G_s^{(2)}(-, 1)$. Note if $g_s^{(1)}$ was linear, then $g_s^{(1)} = G_s^{(2)}(-, t)$ for all t .

This completes the proof that the map $\mathrm{GL}_n(\mathbb{R}) \hookrightarrow \mathrm{Emb}_0^{\mathrm{Diff}}(\mathbb{R}^n, \mathbb{R}^n)$ is a weak equivalence. We remark that if all g_s were diffeomorphisms, i.e. surjective, then so are all $G_s^{(1)}(-, t)$ and $G_s^{(2)}(-, t)$ and thus the same argument tells us that $\mathrm{GL}_n(\mathbb{R}) \hookrightarrow \mathrm{Diff}(n)$ is a weak equivalence. \square

2.2. Topological self-embeddings of \mathbb{R}^n . We next repeat the entire exercise in the topological setting. We want to show that the inclusion

$$\mathrm{Top}(n) \hookrightarrow \mathrm{Emb}_0^{\mathrm{Top}}(\mathbb{R}^n, \mathbb{R}^n)$$

is a weak equivalence. As before it suffices to find a lift

$$\begin{array}{ccc} S^k & \longrightarrow & \mathrm{Top}(n) \\ \downarrow & \nearrow g_s^{(2)} & \downarrow \\ D^{k+1} & \xrightarrow{g_s} & \mathrm{Emb}_0^{\mathrm{Top}}(\mathbb{R}^n, \mathbb{R}^n) \end{array}$$

after homotoping the diagram. That is, we want to deform the family g_s , $s \in D^{k+1}$ to homeomorphisms staying in homeomorphisms if we already are in homeomorphisms. Here it is helpful to remark that since a topological embedding is a homeomorphism onto its image, it is a homeomorphism if and only if it is surjective.

Our strategy is outlined by the following diagram:

$$\begin{array}{ccc} g_s & & \text{our original family} \\ \downarrow \text{canonical circle jiggling} & & \\ g_s^{(1)} & & \text{image equals an open disk} \\ \downarrow \text{zooming in} & & \\ g_s^{(2)} & & \text{surjective} \end{array}$$

The important technical tool replacing Taylor approximation is the following ‘‘sphere jiggling’’ trick. Let $D_r \subset \mathbb{R}^n$ denote the closed disk of radius r around the origin.

Lemma 2.5. *Fix $a < b$ and $c < d$ in $(0, \infty)$. Suppose we have $f, h \in \mathrm{Emb}_0^{\mathrm{Top}}(\mathbb{R}^n, \mathbb{R}^n)$ with $h(\mathbb{R}^n) \subset f(\mathbb{R}^n)$ and $h(D_b) \subset f(D_c)$. Then there exists an isotopy ϕ_t of \mathbb{R}^n such that*

- (i) $\phi_0 = \text{id}$,
- (ii) $\phi_1(h(D_b)) \supset f(D_c)$,
- (iii) ϕ_t fixes pointwise $\mathbb{R}^n \setminus f(D_d)$ and $h(D_a)$.

This is continuous in f, h and a, b, c, d .

Proof. Since $h(\mathbb{R}^n) \subset f(\mathbb{R}^n)$, it suffices to work in f -coordinates. Our isotopy will be compactly supported in these coordinates, so we can extend by the identity to the complement of $f(\mathbb{R}^n)$ in \mathbb{R}^n .

We will now define some subsets in f -coordinates and invite the reader to look at Figure 2. Let b' be the radius of the largest disk contained in $h(D_b)$ (in f -coordinates, remember) and a' the radius of the largest disk contained in $h(D_a)$. Our first attempt for an isotopy is to make ϕ_t piecewise-linearly scale the radii between a and d such that c moves to b' . This satisfies (i), (ii) and fixes $\mathbb{R}^n \setminus f(D_d)$. In words it “pulls $h(D_b)$ over $f(D_c)$. It might not fix $h(D_a)$.”

This can be solved by a trick: we conjugate with a homeomorphism, described in h -coordinates as follows: piecewise-linearly radially scale between 0 and b by moving the radius a to radius a'' , where a'' is the radius of the largest disk contained in $f(D_{a'})$. In words, we temporarily decrease the size of $h(D_a)$ to be contained in $f(D_{a'})$, do our previous isotopy, and restore $h(D_a)$ to its original shape.

The continuity of this construction depends on the continuity of b', a' and a'' , which we leave to the reader as an exercise in the compact-open topology. \square

Theorem 2.6. *The inclusion*

$$\text{Top}(n) \hookrightarrow \text{Emb}_0^{\text{Top}}(\mathbb{R}^n, \mathbb{R}^n)$$

is a weak equivalence.

Proof. Following the strategy outlined before, we have two steps.

- (i) Our first step involves making the image of g_s into a (possibly infinite) open disk. Let $R_s(r)$ be the piecewise linear function $[0, \infty) \rightarrow [0, \infty)$ sending $i \in \mathbb{N}_0$ to the radius of the largest disk contained in $g_s(D_i)$. Then we can construct an element of $\text{Emb}_0^{\text{Top}}(\mathbb{R}^n, \mathbb{R}^n)$ given in radial coordinates by $h_s(r, \varphi) = (R_s(r), \varphi)$. This satisfies $h_s(\mathbb{R}^n) \subset g_s(\mathbb{R}^n)$, $h_s(D_i) \subset g_s(D_i)$ for all $i \in \mathbb{N}_0$ and has image an open disk. It is continuous in s .

Our goal is to deform h_s to have the same image as g_s in infinitely many steps. For $t \in [0, 1/2]$ we use the lemma to push $h_s(D_1)$ to contain $g_s(D_1)$ while fixing $g_s(D_2)$. For $t \in [1/2, 3/4]$ we use the lemma to push the resulting image of $h_s(D_2)$ to contain $g_s(D_2)$ while fixing $g_s(D_3)$ and the resulting image of $h_s(D_1)$, etc. These infinitely many steps converge to an embedding since on each compact only finitely many steps are not the identity. The result is a family $H_s(-, t)$ in $\text{Emb}_0^{\text{Top}}(\mathbb{R}^n, \mathbb{R}^n)$ such that $H_s(-, 1)$ has the same image as g_s . It is continuous in s since h_s is. So step (i) does this:

$$G_s^{(1)}(x, t) := H_s(H_s(-, 1)^{-1}g_s(x), 1 - t)$$

For $t = 0$, this is simply $g_s(x)$. For $t = 1$, this is $H_s(-, 1)^{-1}(g_s(x))$, which we denote by $g_s^{(1)}$ and has the same image as $h_s(x)$, i.e. a possibly infinite open disk. Note that if g_s were surjective, then so $G_s^{(1)}(x, t)$ for all t .

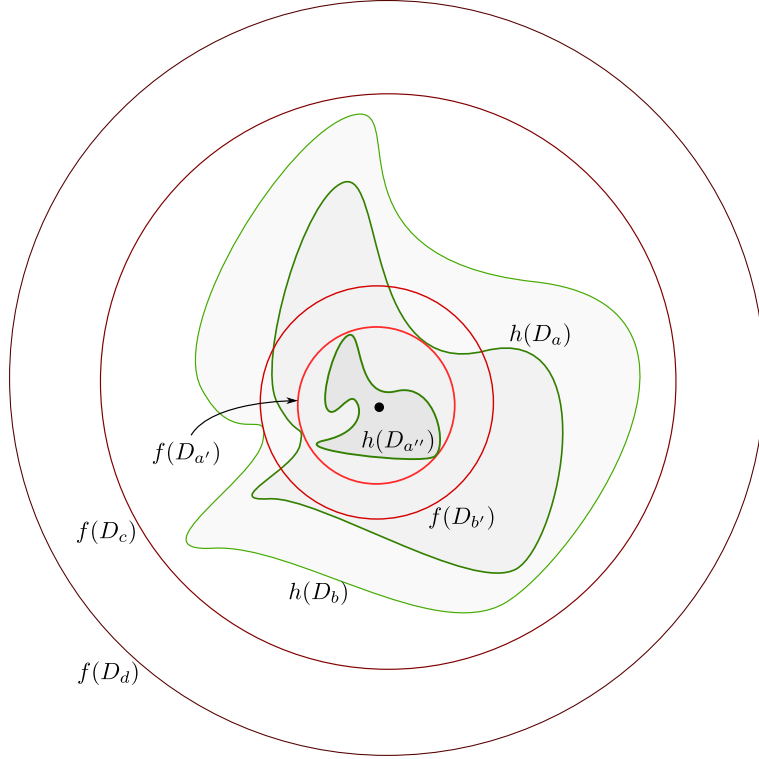


FIGURE 2. The disks appearing in Lemma 2.5.

- (ii) There is a piecewise-linear radial isotopy K_s moving $h_s(x)$ to the identity. It is given by moving the values of R_s at each integer i to i . We set

$$G_s^{(2)}(x, t) := K_s(-, 1 - t)^{-1} g_s^{(1)}(x)$$

so that for $t = 0$ we have get $g_s^{(1)}$ and for $t = 1$ we get $g_s^{(2)}$ with image \mathbb{R}^n . Note that if $g_s^{(1)}$ were surjective, then so $G_s^{(2)}(x, t)$ for all t .

□

3. LECTURE 2: MICROBUNDLE TRANSVERSALITY

In this lecture we describe a notion of transversality for topological manifolds, and prove a transversality result for topological manifolds. We start by recalling the situation for smooth manifolds.

3.1. Smooth transversality. The theory of transversality for smooth manifolds requires us to discuss tangent bundles and differentials. Every smooth manifold N has a tangent bundle TN . A smooth submanifold X of a smooth manifold of N is a closed subset such that there for each $x \in X$ there exists a chart U of N containing x such that the pair $(U, U \cap X)$ is diffeomorphic to $(\mathbb{R}^n, \mathbb{R}^x)$. The tangent bundle TX of a smooth submanifold $X \subset N$ can be

canonically identified with a subbundle of $TN|_X$. This identification is a special case of the derivative of a smooth map $f: M \rightarrow N$, which is a map $Tf: TM \rightarrow TN$ of vector bundles.

Definition 3.1. Let M, N be smooth manifolds, $X \subset N$ a smooth submanifold and $f: M \rightarrow N$ be a smooth map. Then f is *transverse* to X if for all $x \in X$ and $m \in f^{-1}(x)$ we have that $Tf(TM_m) + TX_x = N_x$.

Some remarks about this definition:

- If $X = \{x\}$, then f is transverse to X if and only if for each $m \in f^{-1}(x)$ the differential $Tf: TM_m \rightarrow TN_x$ is surjective. In this case m is said to be a *regular point* and x an *regular value*. Sard's lemma says that regular values are dense.
- The *implicit function theorem* says that if f is transverse to X then $f^{-1}(X) \subset M$ is a smooth submanifold. It will be of codimension $n - x$, hence of dimension $m + x - n$.
- If f is the inclusion of a submanifold, this definition simplifies: two smooth submanifolds M and X are transverse if $TM_x + TX_x = TN_x$ for all $x \in M \cap X$. The implicit function theorem then says that we can find a chart U near m such that $N \cap U$ and $X \cap U$ in this chart are given by two affine planes intersecting generically (that is, in an $m + x - n$ -dimensional affine plane).
- Note that if $\dim M + \dim X < \dim N$ then f and X are transverse if and only if $f(M)$ and X are disjoint.

Lemma 3.2. *Every smooth map $f: M \rightarrow N$ can be approximated by a smooth map transverse to X .*

Proof. To do induction over charts in Step 3, we actually need to prove a strongly relative version. That is, we assume we are given closed subsets $C_{\text{done}}, D_{\text{todo}} \subset M$ and open neighborhoods $U_{\text{done}}, V_{\text{todo}} \subset M$ of $C_{\text{done}}, D_{\text{todo}}$ respectively, such that f is already transverse to X on U_{done} (note that $C_{\text{done}} \cap D_{\text{todo}}$ could be non-empty). See Figure 3. It will be helpful to let $r := n - x$ denote the codimension of X .

Then we want to make f transverse on a neighborhood of $C_{\text{done}} \cup D_{\text{todo}}$ without changing it on a neighborhood of $C_{\text{done}} \cup (M \setminus V_{\text{todo}})$. We will ignore the smallness of the approximation.

Step 1: M open in \mathbb{R}^m , $X = \{0\}$, $N = \mathbb{R}^r$: We first prove that the subset of $C^\infty(M, \mathbb{R}^r)$

which consists of smooth functions $M \rightarrow \mathbb{R}^r$ that are transverse to $\{0\}$ is open and dense (the C^∞ -topology on spaces of smooth functions is the one in which a sequence converges if and only if all derivatives converge on compacts). This is analysis and the only place where smoothness plays a role.

Recall that f is transverse to 0 at a point $m \in M$ if either (i) $f(m) \neq 0$, or (ii) $f(m) = 0$ and $Tf_m: TM_m \rightarrow T\mathbb{R}_0^r$ is surjective. Openness follows from the fact that (i) and (ii) are open conditions. For density, we use Sard's lemma [Hir94, Theorem 3.1.3], which says that for any smooth function the regular values are dense in the target. Thus for every $f \in C^\infty(M, \mathbb{R}^r)$ there is a sequence of $x_k \in \mathbb{R}^r$ of regular values of $f: M \rightarrow \mathbb{R}^r$ converging to 0. Then

$$f_k := f - x_k$$

is a sequence of functions transverse to $\{0\}$ converging to f .

This is not a relative version yet. Fix a smooth function $\eta: M \rightarrow [0, 1]$ such that η is 0 on $C_{\text{done}} \cup (M \setminus V_{\text{todo}})$ and 1 on a neighborhood of $D_{\text{todo}} \setminus U_{\text{done}}$. Then consider

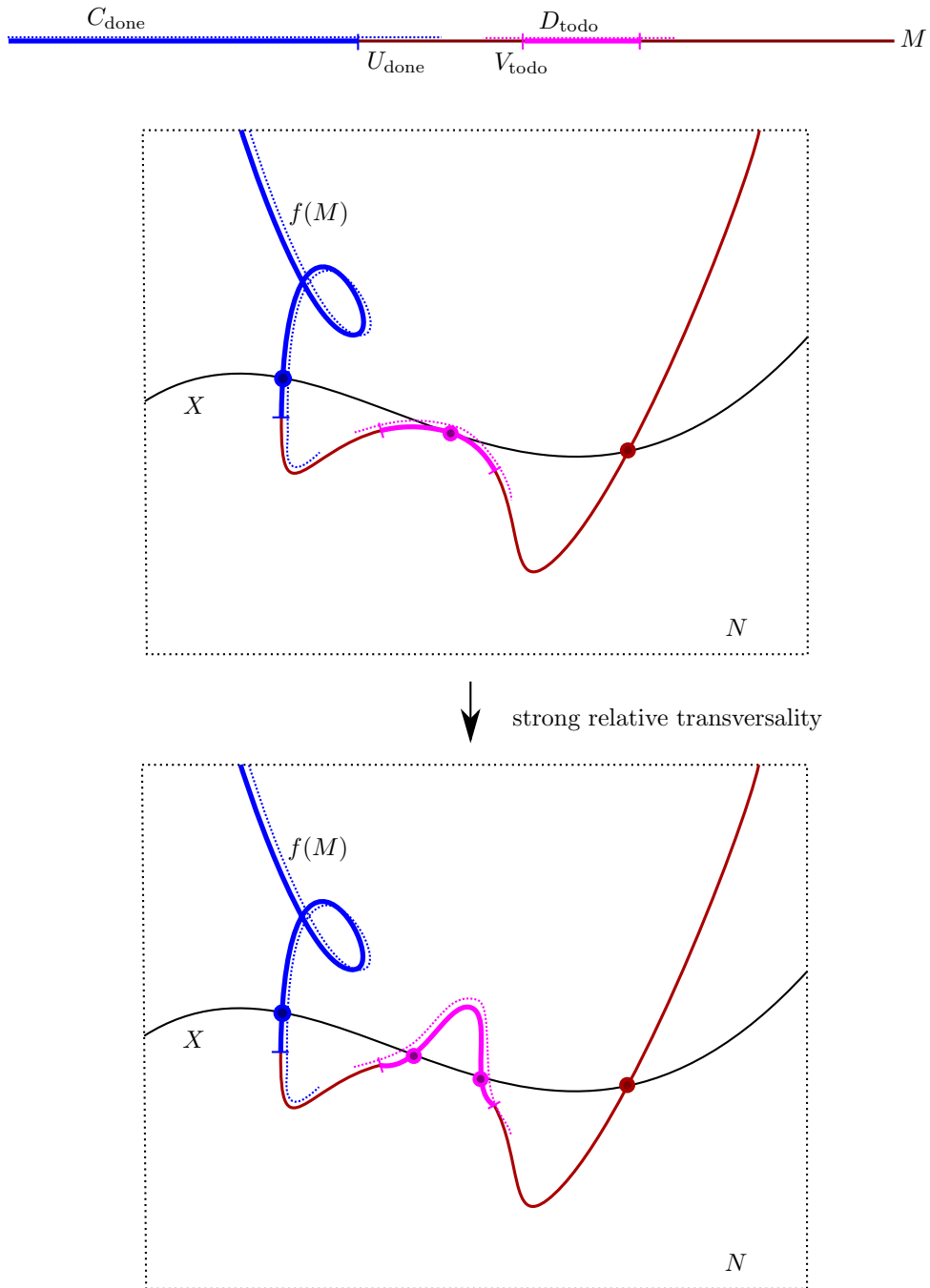


FIGURE 3. The data for the strongly relative version in Lemma 3.2 and the result of the strong relative transversality.

the smooth functions

$$\tilde{f}_k := \eta f_k + (1 - \eta)f$$

For k sufficiently large, this is transverse to $\{0\}$ on a neighborhood of $C_{\text{done}} \cup D_{\text{todo}}$, by openness of the condition of being transverse to $\{0\}$.

Step 2: M open in \mathbb{R}^m , ν_X trivializable: Since ν_X is trivializable, we may take a trivialized tubular neighborhood $\mathbb{R}^r \times X$ in N , and substitute

- $M' = f^{-1}(\mathbb{R}^r \times X)$,
- $C'_{\text{done}} = C_{\text{done}} \cup f^{-1}(X \times (\mathbb{R}^r \setminus \text{int}(D^r))) \cap f^{-1}(\mathbb{R}^r \times X)$,
- $D'_{\text{todo}} = D_{\text{todo}} \cap f^{-1}(\mathbb{R}^r \times X)$.

This reduces to the case $N = \mathbb{R}^r \times X$, and then $f: M \rightarrow N = \mathbb{R}^r \times X$ is transverse to X if and only if $\bar{f} := \pi_1 \circ f: M \rightarrow \mathbb{R}^r \times X \rightarrow \mathbb{R}^r$ is transverse to $\{0\}$. This may be achieved by Step 1.

Step 3: General case: This will be an induction over charts. Since our manifolds are always assumed paracompact we can find a covering U_α of X so that each $\nu_X|_{U_\alpha}$ is trivializable. We can then find a locally finite collection of charts $\{\phi_\beta: \mathbb{R}^m \supset V_\beta \hookrightarrow M\}$ covering V_{todo} , such that (i) $D^m \subset V_\beta$, (ii) $D_{\text{todo}} \subset \bigcup_\beta \phi_\beta(D^m)$ and (iii) for all β there exists an α with $f(\phi_\beta(V_\beta)) \subset U_\alpha$.

Order the β , and write them as $i \in \mathbb{N}$ from now on. By induction one then constructs a deformation to f_i transverse on some open U_i of $C_i := C \cup \bigcup_{j \leq i} \phi_j(D^m)$. The induction step from i to $i + 1$ uses step (2) using the substitution $\bar{M} = V_{i+1}$, $C_{\text{done}} = \phi_{i+1}^{-1}(C_i)$, $U_{\text{done}} = \phi_{i+1}^{-1}(U_i)$, $D_{\text{todo}} = D^m$ and V_{todo} is $\text{int}(2D^m)$.

Then a deformation is given by putting the deformation from f_i to f_{i+1} in the time period $[1 - 1/2^i, 1 - 1/2^{i+1}]$. This is continuous as $t \rightarrow 1$ since the cover was locally finite. \square

3.2. Microbundles. Let us reinterpret smooth transversality in terms of normal bundles. Recall that $f: M \rightarrow N$ was transverse to a smooth submanifold $X \subset N$ if for all $x \in X$ and $m \in f^{-1}(x)$ we have that $Tf(TM_m) + TX_x = TN_x$. This is equivalent to the statement that $Tf: TM_m \rightarrow \nu_x := TN_x/TX_x$ is surjective. The vector bundle $\nu := TN|_X/TX$ over X is the so-called *normal bundle*.

In the topological world the notion of a vector bundle is replaced by that of a microbundle, due to Milnor [Mil64].

Definition 3.3. An n -dimensional microbundle ξ over a space B is a triple $\xi = (X, i, p)$ of a space X with maps $p: X \rightarrow B$ and $i: B \rightarrow X$ such that

- $p \circ i = \text{id}$
- for each $b \in B$ there exists open neighborhoods $U \subset B$ of b and $V \subset p^{-1}(U) \subset X$ of $i(b)$ and a homeomorphism $\phi: \mathbb{R}^n \times U \rightarrow V$ such that $\{0\} \times U \rightarrow \mathbb{R}^n \times U \rightarrow V$ coincides with i and $\mathbb{R}^n \times U \rightarrow V \rightarrow B$ coincides with the projection to U . More precisely, the following diagrams should commute

$$\begin{array}{ccc} \mathbb{R}^n \times U & \xrightarrow{\phi} & V \\ \uparrow \iota & & \uparrow i|_U \\ U & \xlongequal{\quad} & U \end{array} \qquad \begin{array}{ccc} \mathbb{R}^n \times U & \xrightarrow{\phi} & V \\ \pi_2 \downarrow & & \downarrow p \\ U & \xlongequal{\quad} & U \end{array}$$

Two n -dimensional microbundles $\xi = (X, i, p)$, $\xi' = (X', i', p')$ over B are *equivalent* if there are neighborhoods W of $i(B)$ and W' of $i'(B)$ and a homeomorphism $W \rightarrow W'$ compatible with all the data.

Example 3.4. If $\Delta: M \rightarrow M \times M$ denotes the diagonal, and $\pi_2: M \times M \rightarrow M$ the projection on the second factor, then $(M \times M, \Delta, \pi_2)$ is the *tangent microbundle* of M .

To show it is an m -dimensional microbundle near b , pick a chart $\psi: \mathbb{R}^n \rightarrow M$ such that $b \in \psi(\mathbb{R}^n)$. Since the condition on the existence of the homeomorphism ϕ in the definition of a microbundle is local, it suffices to prove that the diagonal in \mathbb{R}^n has one of these charts. Indeed, we can take $U = \mathbb{R}^n$, $V = \mathbb{R}^n \times \mathbb{R}^n$ and $\phi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ given by $(x, y) \mapsto (x + y, y)$.

Example 3.5. Every vector bundle is a microbundle. If M is a smooth manifold, the tangent microbundle is equivalent to the tangent bundle. This is a consequence of the tubular neighborhood theorem.

Kister's theorem allows us to describe these microbundles in more familiar terms. To give this description, we use that microbundles behave in many respects like vector bundles. For example, any microbundle over a paracompact contractible space B is trivial, i.e. equivalent to $(\mathbb{R}^n \times B, \iota_0, \pi_2)$. This is Corollary 3.2 of [Mil64]. The following appears in [Kis64] and is a consequence of Theorem 1.16.

Definition 3.6. An \mathbb{R}^n -*bundle* over a space B is a bundle with fibers \mathbb{R}^n and transition functions in the topological group consisting of homeomorphisms of \mathbb{R}^n fixing the origin.

We say that two \mathbb{R}^n -bundles ξ_0, ξ_1 over B are *concordant* if there is an \mathbb{R}^n -bundle ξ over $B \times I$ that for $i \in \{0, 1\}$ restricts to ξ_i over $B \times \{i\}$.

Theorem 3.7 (Kister-Mazur). *Every n -dimensional microbundle $\xi = (E, i, p)$ over a sufficiently nice space (e.g. locally finite simplicial complex or a topological manifold) is equivalent to an \mathbb{R}^n -bundle. This bundle is unique up to isomorphism (in fact concordance).*

Proof. Let us assume that the base is a locally finite simplicial complex. The total space of our \mathbb{R}^n -bundle \mathcal{E} will be a subset of the total space E of the microbundle, and we will find it inductively over the simplices of B .

We shall content ourselves by proving the basic induction step, by showing how to extend \mathcal{E} from $\partial\Delta^i$ to Δ^i , see Figure 4. So suppose we are given a \mathbb{R}^n -bundle $\mathcal{E}_{\partial\Delta^i}$ inside $\xi|_{\partial\Delta^i}$. For an inner collar $\partial\Delta^i \times [0, 1]$ of $\partial\Delta^i$ in Δ^i , the local triviality allows us to extend it to $\mathcal{E}_{\partial\Delta^i \times [0, 1]}$ inside $\xi|_{\partial\Delta^i \times [0, 1]}$. Since Δ^i is contractible, we can trivialize the microbundle $\xi|_{\Delta^i}$ and in particular it contains a trivial \mathbb{R}^n bundle $\mathcal{E}_{\Delta^i} \cong \mathbb{R}^n \times \Delta^i$. By shrinking the \mathbb{R}^n , we may assume that its restriction to $\partial\Delta^i \times [0, 1]$ is contained in $\mathcal{E}_{\partial\Delta^i \times [0, 1]}$. Thus for each $x \in \partial\Delta^i$, we get a map $\phi_x: [0, 1] \rightarrow \text{Emb}_0(\mathbb{R}^n, \mathbb{R}^n)$. Using the canonical isotopy proved by Kister's theorem 1.16, we can isotope this family continuously in x to $\tilde{\phi}_x$ satisfying (i) $\tilde{\phi}_x(0)$ is a homeomorphism, (ii) $\tilde{\phi}_x(1) = \phi_x(1)$. Thus the replacement of $\mathcal{E}_{\Delta^i}|_{\Delta^i \times [0, 1]} \subset \mathcal{E}_{\partial\Delta^i \times [0, 1]}$ by $\tilde{\phi} \circ \phi^{-1}(\mathcal{E}_{\Delta^i}|_{\Delta^i \times [0, 1]})$, is an extension of $\mathcal{E}_{\partial\Delta^i}$ to Δ^i . \square

Remark 3.8. Warning: an \mathbb{R}^n -bundle need *not* contain a disk bundle, in contrast with the case of vector bundles [Bro66]. Even if there exists one, it does not need to be unique [Var67].

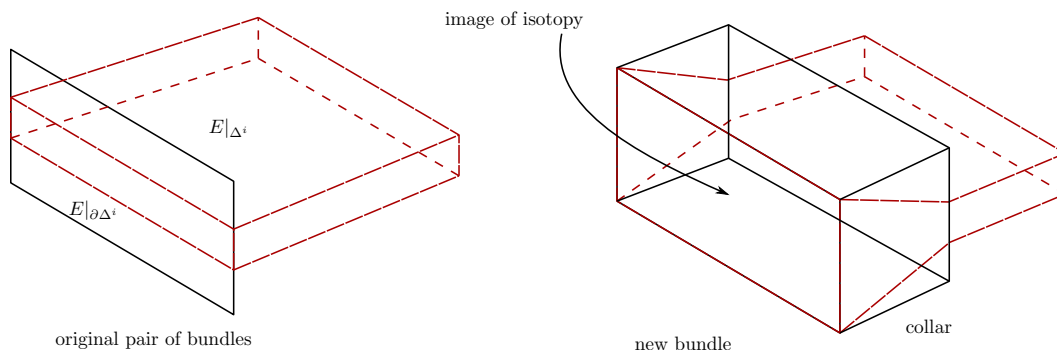


FIGURE 4. The extension in the proof of the Kister-Mazur theorem.

3.3. Locally flat submanifolds and normal microbundles. The appropriate generalization of a smooth submanifold to the setting of topological manifolds is given by the following definition.

Definition 3.9. A *locally flat submanifold* X of a topological manifold M is a closed subset X such that for each $x \in X$ there exists an open subset U of M and a homeomorphism from U to \mathbb{R}^m which sends $U \cap X$ homeomorphically onto $\mathbb{R}^x \subset \mathbb{R}^m$.

Locally flat submanifolds are well behaved: every locally flat submanifold of codimension 1 admits a bicollar, the Schoenflies theorem for locally flat S^{n-1} 's in S^n says that any locally flat embedded S^{n-1} in S^n has as a complement two components whose closures are homeomorphic to disks [Bro60]. Moreover, there is an isotopy extension theorem for locally flat embeddings [EK71].

Exercise 3.1. Prove that every locally flat submanifold of codimension 1 admits a bicollar. Use this in a different version of the proof of Theorem 1.22, by noting that all M_i constructed have locally flat boundary ∂M_i in M . Generalize Theorem 1.22 to the relative case: if M is of dimension ≥ 6 and contains a codimension zero submanifold A with handle decomposition and locally flat boundary ∂A , then we may extend the handle decomposition of A to one of M .

Remark 3.10. This is not the only possible definition: one could also define a “possibly wild” submanifold to be the image of a map $f: X \rightarrow M$, where X is a topological manifold and f is a homeomorphism onto its image.

Examples of such possibly wild submanifolds include the Alexander horned sphere in S^3 and the Fox-Artin arc in \mathbb{R}^3 . These exhibit what one may consider as pathological behavior: the Schoenflies theorem fails for the Alexander horned sphere (it is not the case that the closure of each component of its complement is homeomorphic to a disk). The Fox-Artin arc has a complement which is not simply-connected, showing that there is no isotopy extension theorem for wild embeddings.

See [DV09] for more results about wild embeddings. We do note that there exists a theory of “taming” possibly wild embeddings in codimension 3, see [Las76, Appendix]. This theory

may be summarized by saying that the spaces (really simplicial sets) of locally flat and possibly wild embeddings are weakly equivalent.

Definition 3.11. A *normal microbundle* ν for a locally flat submanifold $X \subset N$ is a $(n-x)$ -dimensional microbundle $\nu = (E, i, p)$ over X together with an embedding of a neighborhood U in E of $i(X)$ into N . The composite $X \hookrightarrow U \hookrightarrow N$ should be the identity.

Does a locally flat submanifold always admit a normal microbundle? This is true in the smooth case, as a consequence of the tubular neighborhood theorem.

Lemma 3.12. *Any smooth submanifold X of a smooth manifold N has a normal microbundle.*

Remark 3.13. In fact, one can define *smooth microbundles*, requiring all maps to the smooth and replacing homeomorphisms with diffeomorphisms. A smooth submanifold has a unique smooth normal microbundle.

However, the analogue of Lemma 3.12 is not true in the topological case, and there is an example of Rourke-Sanderson in the PL case [RS67]. However, they do exist after stabilizing by taking a product with \mathbb{R}^s [Bro62]. The uniqueness statement uses the notion of concordance. We say that two normal microbundles ν_0, ν_1 over $X \subset N$ are *concordant* if there is a normal bundle ν over $X \times I \subset N \times I$ restricting for $i \in \{0, 1\}$ to ν_i on $X \times \{i\}$.

Theorem 3.14 (Brown). *If $X \subset N$ is a locally flat submanifold, then there exists an $S \gg 0$ depending only on $\dim X$ and $\dim N$, such that X has a normal microbundle in $N \times \mathbb{R}^s$ if $s \geq S$, which is unique up to concordance if $s \geq S + 1$.*

Remark 3.15. By the existence and uniqueness of collars, normal microbundles do exist in codimension one. Kirby-Siebenmann proved they exist and are unique in codimension two (except when the ambient dimension is 4) [KS75], the case relevant to topological knot theory. Finally, in dimension 4 normal bundles always exist by Freedman-Quinn [FQ90, Section 9.3]. One can also relax the definitions and remove the projection map but keep a so-called block bundle structure. Normal block bundles exist and are unique in codimension ≥ 5 or ≤ 2 , see [RS70].

3.4. Topological microbundle transversality. We will now describe a notion of transversality for topological manifolds, which generalizes smooth transversality and makes the normal bundles part of the data of transversality.

Definition 3.16. Let $X \subset N$ be a locally flat submanifold with normal microbundle ξ . Then a map $f: M \rightarrow N$ is said to be *microbundle transverse to ξ (at ν)* if

- $f^{-1}(X) \subset M$ is a locally flat submanifold with a normal microbundle ν in M ,
- f gives an open topological embedding of a neighborhood of the zero section in each fiber of ν into a fiber of ξ .

See Figure 5 for an example. We now prove that this type of transversality can be achieved by small perturbations, as we did in Lemma 3.2 for smooth transversality.

Remark 3.17. One can also define smooth microbundle transversality to a smooth normal microbundle. This differs from ordinary transversality in the sense that the smooth manifold

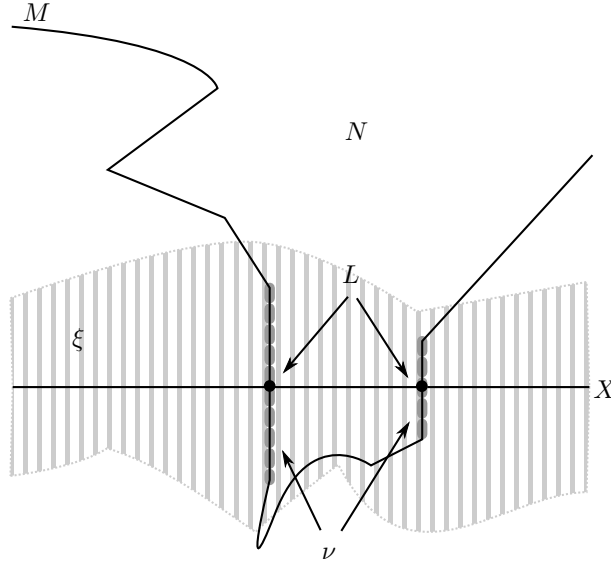


FIGURE 5. An example of microbundle transversality for $f: M \rightarrow N$ given by the inclusion of a submanifold.

has to line up with normal microbundle near the manifold X , i.e. the difference between the intersection “at an angle” of Figure 3 and the “straight” intersection of Figure 5. Smooth microbundle transversality implies ordinary transversality, and any smooth transverse map can be made smooth microbundle transverse. This is why one usually does not discuss the notion of smooth microbundle transversality.

Remark 3.18. There are other notions of transversality that may be more well-behaved; in particular there is Marin’s stabilized transversality [Mar77] and block transversality [RS67]. A naive local definition is known to be very badly behaved: a relative version is false [Hud69].

The following is a special case of [KS77, Theorem III.1.1].

Theorem 3.19 (Topological microbundle transversality). *Let $X \subset N$ be a locally flat submanifold with normal microbundle $\xi = (E, i, p)$. If $m + x - n \geq 6$ (the expected dimension of $f^{-1}(X)$), then every map $f: M \rightarrow N$ can be approximated by a map which microbundle transverse to ξ .*

Proof. The steps of our proof are the same as those in Lemma 3.2. Again, we actually need to prove a strongly relative version. That is, we assume we are given $C_{\text{done}}, D_{\text{todo}} \subset M$ closed and $U_{\text{done}}, V_{\text{todo}} \subset M$ open neighborhoods of $C_{\text{done}}, D_{\text{todo}}$ respectively such that f is already microbundle transverse to ξ at ν_{done} over a submanifold $L_{\text{done}} := f^{-1}(X) \cap U_{\text{done}}$ in U_{done} (note that $C_{\text{done}} \cap D_{\text{todo}}$ could be non-empty). We invite the reader to look at Figure 3 again. It will be helpful to let $r := n - x$ denote the codimension of X .

Then we want to make f microbundle transverse to ξ at some ν on a neighborhood of $C_{\text{done}} \cup D_{\text{todo}}$ without changing it on a neighborhood of $C_{\text{done}} \cup (M \setminus V_{\text{todo}})$. We will also

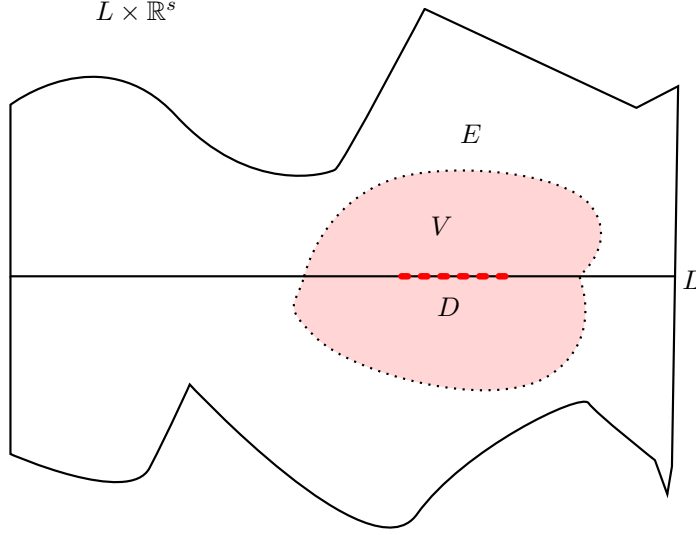


FIGURE 6. The data in the local product structure theorem.

ignore the smallness of the approximation, as it is a theorem that a strongly relative result *always* implies an ϵ -small result, see Appendix I.C of [KS77].

Step 1: M open in \mathbb{R}^m , $X = \{0\}$, ξ is a product, $N = E = \mathbb{R}^r$: We want to apply the relative version of smooth transversality (with the small smooth microbundle transversality improvement mentioned in Remark 3.17). To do this we need to find a smooth structure Σ on M such that for some open neighborhood W_Σ of $f^{-1}(0) \cap C_{\text{done}} \subset M$, the microbundle $\nu_{\text{done}} \cap W_\Sigma$ over $L_{\text{done}} \cap W_\Sigma$ is smooth and $f: M_\Sigma \rightarrow \mathbb{R}^n$ is transverse at $\nu_{\text{done}} \cap W_\Sigma$ to 0 near C_{done} .

This uses a version of the product structure theorem. The version we stated before said that concordance classes of smooth structures on $M \times \mathbb{R}$ are in bijection to concordance classes of smooth structures on M . We need a local version, specializing [KS77, Theorem I.5.2]:

Suppose one has a topological manifold L of dimension ≥ 6 , an open neighborhood E of $L \times \{0\} \subset L \times \mathbb{R}^s$, a smooth structure Σ on E , $D \subset L \times \{0\}$ closed and $V \subset E$ an open neighborhood of D . Then there exists a concordance of smooth structures on E rel $(E \setminus V)$ from Σ to a Σ' that is a product near D . See Figure 6.

We want to substitute the data (L, E, s, Σ, D, V) of this theorem by the data $(L_{\text{done}}, E(\nu_{\text{done}}), r, \Sigma', L_{\text{done}} \cap C_{\text{done}}, V')$ with Σ' and V' to be defined. Thus here we get the condition that $\dim(L_{\text{done}}) = m - r = m + x - n \geq 6$. For this substitution to make sense, we must have that E_{done} is an open subset of $L_{\text{done}} \times \mathbb{R}^r$, which comes from the open inclusion $(p, f): E(\nu_{\text{done}}) \hookrightarrow L_{\text{done}} \times \mathbb{R}^r$. In terms of the latter coordinates f is simply the projection $\pi_2: L_{\text{done}} \times \mathbb{R}^r \rightarrow \mathbb{R}^r$. Since $E(\nu_{\text{done}}) \subset M \subset \mathbb{R}^m$, it inherits the standard smooth structure. The set V' will be an open neighborhood of L_{done} in $E(\nu_{\text{done}})$ with closure also contained in $E(\nu_{\text{done}})$.

Then the application of the local version of the product structure theorem gives us a smooth structure on V' , which can be extended by the standard smooth structure to M since we did not modify it outside V' . In this smooth structure L_{done} is smooth and ν_{done} is just the product with \mathbb{R}^n . This implies that f is now smooth, as it is given by the projection $(L_{\text{done}})_{\Sigma} \times \mathbb{R}^r \rightarrow \mathbb{R}^r$.

To finish this step, outside a small neighborhood of L_{done} we smooth f near D_{todo} without modifying outside V_{todo} and then apply a relative version of transversality with the same constraints on where we make the modifications.

Step 2: M open in \mathbb{R}^m , ξ trivializable: Since ξ is trivializable we may assume $E(\xi)$ contains $\mathbb{R}^r \times X$. If we substitute

- $M' = f^{-1}(\mathbb{R}^r \times X)$,
- $C'_{\text{done}} = (C_{\text{done}} \cup f^{-1}(X \times (\mathbb{R}^r \setminus \text{int}(D^n)))) \cap f^{-1}(\mathbb{R}^r \times X)$,
- $D'_{\text{todo}} = D_{\text{todo}} \cap f^{-1}(\mathbb{R}^r \times X)$,

we reduce to the case where ξ is a product and $Y = \mathbb{R}^r \times X$.

Then we have that $f: M \rightarrow Y$ is given by $(f_1, f_2): M \rightarrow \mathbb{R}^r \times X$. Consider the map $f_1: M \rightarrow \mathbb{R}^r$. Then $(f_1)^{-1}(\{0\}) \cap U_{\text{done}} = f^{-1}(X) \cap U_{\text{done}}$ is a locally flat submanifold L_{done} with normal microbundle and f_1 embeds a neighborhood of the 0-section of the fibers of ν_{done} into \mathbb{R}^r , the fiber of projection to a point. By the previous step we thus can make f_1 microbundle transverse to ξ near $C_{\text{done}} \cup D_{\text{todo}}$ by a small perturbation to some f'_1 , while fixing it on a neighborhood of $C_{\text{done}} \cup (M \setminus V_{\text{todo}})$.

A minor problem now appears when we add back in the component f_2 : even though $f' := (f'_1, f_2)$ has $(f')^{-1}(X)$ a locally flat submanifold with normal bundle, f' may not embed neighborhoods of the 0-section of fibers of this normal bundle into fibers of $\mathbb{R}^r \times X \rightarrow \mathbb{R}^r$. This would be resolved if we precomposed f_2 by a map that near $C_{\text{done}} \cup D_{\text{todo}}$ collapses a neighborhood of the 0-section into the 0-section in a fiber-preserving way, extending by the identity outside a closed subset containing this neighborhood. Such a map can easily be found, see e.g. Lemma III.1.3 of [KS77].

Step 3: General case: This will be an induction over charts, literally the same as Step 3 for smooth transversality. We can find a covering U_{α} of X so that each $\xi|_{U_{\alpha}}$ is trivializable. Since X is paracompact (by our definition of topological manifold), we can find a locally finite collection of charts $\{\phi_{\beta}: \mathbb{R}^m \supset V_{\beta} \hookrightarrow M\}$ covering V_{todo} , such that (i) $D^m \subset V_{\beta}$, (ii) $D_{\text{todo}} \subset \bigcup_{\beta} \phi_{\beta}(D^m)$ and (iii) for all β there exists an α with $f(\phi_{\beta}(V_{\beta})) \subset U_{\alpha}$.

Order the β , and write them as $i \in \mathbb{N}$ from now on. By induction one then constructs a deformation to f_i transverse on some open U_i of $C_i := C \cup \bigcup_{j \leq i} \phi_j(D^m)$. The induction step from i to $i+1$ uses step (2) using the substitution $\bar{M} = V_{i+1}$, $C_{\text{done}} = \phi_{i+1}^{-1}(C_i)$, $U_{\text{done}} = \phi_{i+1}^{-1}(U_i)$, $D_{\text{todo}} = D^m$ and V_{todo} is $\text{int}(2D^m)$.

Then a deformation is given by putting the deformation from f_i to f_{i+1} in the time period $[1 - 1/2^i, 1 - 1/2^{i+1}]$. This is continuous as $t \rightarrow 1$ since the cover was locally finite. \square

Remark 3.20. We could have used the local version of the product structure version in place of the ordinary product structure theorem and concordance extension in the proof of Theorem 1.22.

Exercise 3.2. Prove the case $m + x - n < 0$ of Theorem 3.19, which does not require the local product structure theorem.

If re-examine the proof to see how important the role of the normal bundle ξ to X is, we realize that we only used that X is paracompact and that X is the zero-section of an \mathbb{R}^n -bundle. The microbundle transversality result with these weaker assumptions on X will be used in the next lecture.

Remark 3.21. Theorem 3.19 does *not* prove that if M and X are locally flat submanifolds and X has a normal microbundle ξ , then M can be isotoped to be microbundle transverse to ξ . The reason is that the smoothing of f in step (1) destroys embeddings, as locally flat embeddings are not open.

However, an embedded microbundle transversality result like this is true. The proof in [KS77, Theorem III.1.5] bootstraps from PL manifolds instead, a category of manifolds in which it does not even make sense to talk about openness (PL maps should always be considered as a simplicial set). One additional complication is that finding adapted PL structures requires a result of taming theory, which says that for a topological embedding of a PL manifold of codimension ≥ 3 into a PL manifold, the PL structures on the target can be modified so that the embedding is PL. This is applied to both X and N .

4. LECTURE 3: THE PONTRYAGIN-THOM THEOREM

In this lecture we will state and prove the Pontryagin-Thom theorem for topological manifolds. This is an elementary and important classification result for manifolds, greatly generalized in recent work on cobordism categories, e.g. [GTMW09]. It identifies groups of topological manifolds up to bordism with homotopy groups of a so-called Thom spectrum. The latter is relatively accessible via the tools of stable homotopy theory.

4.1. Thom spectra. We start by defining the Thom spectra mentioned above.

For a topological group G , BG denotes the classifying space for principal G -bundles. This is only a homotopy type, but there is a standard functorial construction called the *bar construction*. It has the property that for nice X (e.g. paracompact), homotopy classes of maps $X \rightarrow BG$ are in bijection with isomorphism classes of principal G -bundles over X . This bijection is given as follows: BG has a principal G -bundle γ_G over it, called the universal bundle, and $f: X \rightarrow BG$ is mapped to $f^*\gamma_G$ over X . In fact, it satisfies a relative classification property for pairs (X, A) , and this in turn implies that the space of classifying maps for a principal G -bundle is weakly contractible.

Example 4.1. If we let G be $\text{Top}(n)$, the topological group of origin-fixing homeomorphisms of \mathbb{R}^n in the compact-open topology, $B\text{Top}(n)$ classifies principal $\text{Top}(n)$ -bundles up to isomorphism. These are in bijection with \mathbb{R}^n -bundles up to isomorphism, our shorthand for fibre bundles with fibre \mathbb{R}^n and structure group $\text{Top}(n)$. The map sends a principal $\text{Top}(n)$ -bundle with total space E to the \mathbb{R}^n -bundle $E \times_{\text{Top}(n)} \mathbb{R}^n$. We will denote the universal \mathbb{R}^n -bundle by γ_n .

Remark 4.2. We can interpret Theorem 3.7 in terms of homotopy theory. A version of Brown representability proves there is a classifying space $B\text{Microb}(n)$ for n -dimensional

topological microbundles. As every \mathbb{R}^n -bundle is an n -dimensional microbundle, there is the map $B\text{Top}(n) \rightarrow B\text{Microb}(n)$ and the Kister-Mazur theorem implies it is a weak equivalence.

Considering the sequence of topological groups $\text{Top}(n)$, we get a sequence of classifying spaces $B\text{Top} = (B\text{Top}(n))_{n \geq 0}$ with connecting maps $B\text{Top}(n) \rightarrow B\text{Top}(n+1)$. We want to generalize this. Warning: the following is not a standard definition, nor an ideal one, but it will be sufficient for our naive approach. For example, it should suffice that B_n and the bundle over it are only defined on a cofinal sequence of n and that the diagram is only homotopy-coherent.

Definition 4.3. A *sequential tangential structure* θ is a sequence of spaces and maps

$$B_0 \xrightarrow{i_0} B_1 \xrightarrow{i_1} \dots$$

with for each B_n has an \mathbb{R}^n -bundle θ_n and isomorphisms $i_n^* \theta_{n+1} \cong \epsilon \oplus \theta_n$.

One way to construct a sequential tangential structure is to give a compatible sequence of maps $B_n \rightarrow B\text{Top}(n)$. Pulling back the universal \mathbb{R}^n -bundle γ_n over $B\text{Top}(n)$ to B_n gives a \mathbb{R}^n -bundle γ_n over B_n satisfying that $i_n^* \gamma_{n+1} \cong \epsilon \oplus \gamma_n$.

Remark 4.4. For standard constructions of classifying spaces, e.g. the bar construction, $B\text{Top}(n) \rightarrow B\text{Top}(n+1)$ is a cofibration and $B\text{Top} = \text{colim}_{n \rightarrow \infty} B\text{Top}(n)$. Thus $B\text{Top}$ has a filtration, and given a map $\theta: B \rightarrow B\text{Top}$, we can pull this back to a filtration of B . The filtration step B_n has a map $B_n \rightarrow B\text{Top}(n)$ and we obtain a sequential tangential structure.

For an \mathbb{R}^n -bundle ξ over a space B , we can construct the *Thom space* $\text{Th}(\xi)$. This is given by taking the fiberwise one-point compactification and collapsing the section at ∞ to a point. If B is compact, this is homeomorphic to the one-point compactification of the total space of ξ . Thus as a set $\text{Th}(\xi)$ is given by the total space of ξ together with a point at ∞ . Note that $\text{Th}(\epsilon^n) \cong S^n \wedge B_+$ and $\text{Th}(\epsilon^k \oplus \xi) \cong S^k \wedge \text{Th}(\xi)$.

Recall that a spectrum E in its most naive form (good enough for our purposes) is a sequence of pointed spaces E_n for $n \geq 0$ together with maps $S^1 \wedge E_n \rightarrow E_{n+1}$.

Definition 4.5. For θ a sequential tangential structure, the *Thom spectrum* $M\theta$ has n th space given by the pointed space $\text{Th}(\theta_n)$ and maps $S^1 \wedge \text{Th}(\theta_n) \rightarrow \text{Th}(\theta_{n+1})$ induced by the isomorphisms $i_n^* \theta_{n+1} \cong \epsilon \oplus \theta_n$.

Example 4.6. For $B = (*)_{n \geq 0}$, we have that $\text{Th}(\theta_n) = S^n$ and $S^1 \wedge S^n \rightarrow S^{n+1}$ the standard isomorphism, so that $M\theta$ is the sphere spectrum \mathbb{S} .

Spectra have stable homotopy groups, given by

$$\pi_n(E) := \text{colim}_{k \rightarrow \infty} \pi_{n+k}(E_k)$$

Example 4.7. The homotopy groups $\pi_n(\mathbb{S})$ are the stable homotopy groups of spheres.

4.2. Cobordism groups. We want to classify topological manifolds up to a certain equivalence relation. We call manifold *closed* if it is compact and we want to stress that it has empty boundary.

Definition 4.8. We say that two n -dimensional closed topological manifolds M, M' are *cobordant* if there is an $(n+1)$ -dimensional compact topological manifold N with $\partial N \cong M \sqcup M'$, called a *cobordism*.

Exercise 4.1. Cobordism is an equivalence relation.

Definition 4.9. The *cobordism group* Ω_n^{Top} of n -dimensional closed topological manifolds is given by the set of homeomorphism classes of n -dimensional compact topological manifolds up to cobordism.

This is an abelian monoid under the operation of disjoint union, and the cobordism $N = M \times I$ shows that $M \sqcup M \sim \emptyset$, so $[M] + [M] = 0$. Thus every element has an inverse and it is in fact an abelian group.

Example 4.10. If $n = 0$, all compact topological n -manifolds are disjoint unions of a finite number of points, hence Ω_0^{Top} is either $\mathbb{Z}/2\mathbb{Z}$ or 0. All cobordisms are disjoint unions of intervals or circles, so the number of points modulo 2 is an invariant. It is easy to see this is a complete invariant, so that we may conclude that $\Omega_0^{\text{Top}} \cong \mathbb{Z}/2\mathbb{Z}$.

If $n = 1$, all compact topological manifolds are disjoint unions of circles. The complement of two disjoint open disks inside a larger disk gives a cobordism from one circle to two circles, so Ω_1^{Top} is generated by $[S^1]$ satisfying $2[S^1] = [S^1]$, i.e. $[S^1] = 0$. We conclude that $\Omega_1^{\text{Top}} \cong 0$.

In these arguments we made some claims about topological 1-manifolds. To make this rigorous, one does an induction over charts (or uses smoothing theory).

Remark 4.11. Since the bordism N admit a relative handle decomposition for $n + 1 \geq 6$ by the improvement of Theorem 1.22 given in Remark 3.1, the equivalence relation of cobordism is generated by those cobordisms obtained by attaching a single handle to $M \times I$.

There is a more general definition of cobordism groups $\Omega_n^{\text{Top}, \theta}$ of topological manifolds equipped with a θ -structure on the stable normal bundle, for θ a sequential tangential structure. To define this, first of all, by a version of the Whitney embedding theorem, every compact n -dimensional topological manifold M can be embedded in an \mathbb{R}^s :

Lemma 4.12. *Every compact topological manifold admits a locally flat embedding into \mathbb{R}^s for $s \gg 0$. This is unique up to ambient isotopy after possibly increasing s .*

Proof. Cover M by charts $\{\phi_i : U_i \hookrightarrow M\}_{i \in I}$ with $U_i \subset \mathbb{R}^n$. By compactness we may assume that I is finite. By paracompactness of M we can find a partition of unity $\{\eta_i : M \rightarrow [0, 1]\}_{i \in I}$ subordinate to $\{\phi_i(U_i)\}_{i \in I}$. Then the map

$$\psi : M \rightarrow \mathbb{R}^{|I|(n+1)}$$

given by $(\eta_i \phi_i^{-1}, \eta_i)_{i \in I}$, is a locally flat embedding. Here we used $\eta_i \phi_i^{-1}$ as shorthand for the map that on U_i equals $\eta_i \phi_i^{-1}$ and is identically 0 elsewhere.

Note that since $\text{supp}(\eta_i) \subset \phi_i(U_i)$, ψ is a well-defined map. It is clearly injective, as $\psi(x) = \psi(y)$ implies that there is an $i \in I$ such that $\eta_i(x) = \eta_i(y) \neq 0$ and then $\frac{1}{\eta_i(x)}(\eta_i(x)\phi_i^{-1}(x)) = \frac{1}{\eta_i(y)}(\eta_i(y)\phi_i^{-1}(y))$ implies $x = y$. We leave it to the reader to check that ψ is locally flat (hint: locally flatness can be checked locally, and if $f : M \rightarrow \mathbb{R}^n$ is a locally flat embedding, then for any continuous function $g : M \rightarrow \mathbb{R}^m$ the map $(f, g) : M \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ is a locally flat embedding).

Given $\phi : M \hookrightarrow \mathbb{R}^s$ and $\phi' : M \hookrightarrow \mathbb{R}^{s'}$ it is easy to give a locally flat isotopy from ϕ to ϕ' in $\mathbb{R}^{s+s'}$. For locally flat submanifolds isotopy implies ambient isotopy by isotopy extension [EK71]. \square

We saw in Theorems 3.14 and 3.7 that this embedding has a normal \mathbb{R}^{s-n} -bundle ν_{s-n} unique up to concordance for s sufficiently large. A θ -structure on ν_{s-n} is a \mathbb{R}^{s-n} -bundle map from ν_{s-n} over M to θ_{s-n} over B_{s-n} , and a θ -structure ζ on M is an equivalence class of normal \mathbb{R}^{s-n} -bundle ν_{s-n} and a θ -structures on ν_{s-n} , up to the equivalence relation given by concordance of all data and increasing s .

If N is an $(n+1)$ -dimensional topological manifold with θ -structure, then its boundary ∂N inherits a θ -structure by picking an interior collar. A cobordism between two n -dimensional compact topological manifolds with θ -structure (M, ζ) and (M', ζ') is an $(n+1)$ -dimensional compact topological manifold with θ -structure (N, ξ) such that $(\partial N, \partial \xi) \cong (M, \zeta) \sqcup (M', \zeta')$.

Example 4.13. If ζ and ζ' are concordant θ -structures on M , then (M, ζ) is cobordant to (M, ζ') . To see this, take the cobordism $M \times I$ with θ -structure given by the concordance.

Definition 4.14. The *cobordism group* $\Omega_n^{\text{Top}, \theta}$ of n -dimensional closed topological manifolds with θ -structure on the stable normal bundle is given by the set of homeomorphism classes of such n -dimensional manifolds, up to cobordism of such $(n+1)$ -dimensional manifolds.

Exercise 4.2. Let $B\text{STop}(n)$ be the classifying space of oriented \mathbb{R}^n -bundles, and STop the associated sequential tangential structure. Understand with help of other sources why Ω_2^{STop} is 0 (giving a complete proof is highly non-trivial).

We can now state the Pontryagin-Thom theorem.

Theorem 4.15 (Pontryagin-Thom for topological manifolds). *There is an isomorphism*

$$\Omega_n^{\text{Top}} \cong \pi_n(M\text{Top})$$

More generally, if $\theta: B \rightarrow B\text{Top}$ is a sequential tangential structure, then

$$\Omega_n^{\text{Top}, \theta} \cong \pi_n(M\theta)$$

This is a generalization to topological manifolds of the classical Pontryagin-Thom theorem for smooth manifolds.

Example 4.16. If we take θ to be $B = (*)_{n \geq 0}$ as before (it may be convenient to replace $*$ by $E\text{Top}(n)$ though), i.e. study topological manifolds with framed stable normal bundles, then the Pontryagin-Thom theorem for topological manifolds tells us that $\Omega_n^{\text{Top}, \text{fr}} \cong \pi_n(\mathbb{S})$. But from the Pontryagin-Thom for smooth manifolds, we also know that $\Omega_n^{\text{Diff}, \text{fr}} \cong \pi_n(\mathbb{S})$. In particular, we conclude that every topological manifold with framed stable normal bundle is cobordant to a smooth manifold. That is, we have proven that

$$\Omega_n^{\text{Top}, \text{fr}} \cong \pi_n(\mathbb{S}) \cong \Omega_n^{\text{Diff}, \text{fr}}$$

Remark 4.17. The result of the previous remark can also be proven by noting that the total space of a normal microbundle is an open subset of $M \times \mathbb{R}^{s-n}$ and inherits a smooth structure from being an open subset of \mathbb{R}^s . Now apply the local product structure theorem. That is, we may also prove:

$$\Omega_n^{\text{Top}, \text{fr}} \cong \Omega_n^{\text{Diff}, \text{fr}} \cong \pi_n(\mathbb{S})$$

This can be reinterpreted in terms of smoothing theory (or for $n = 4$ Lashof-Shaneson-Quinn stabilized smoothing theory [LS71]). Smoothing theory as in Theorem 1.10 says that for $n \neq 4$, the set of smooth structures on M up to concordance is in bijection with

vertical homotopy classes of lifts of $M \rightarrow B\text{Top}$ to BO . The framed version says that smooth structures together with a stable framing up to concordance are in bijection with homotopy classes of lifts of $M \rightarrow B\text{Top}$ to $*$.

Such lifts are also in bijection with the concordance classes of the stable tangent microbundle. These are in turn in bijection with the concordance classes of framings of the stable normal microbundle. This follows from the stable uniqueness of normal bundles, Theorem 3.14. We conclude that each element of $\Omega_n^{\text{Top,fr}}$ is represented by a manifold equipped with a smooth structure and smooth framing of the stable normal bundle, unique up to concordance. Applying the same argument to cobordisms, we see that for $n \neq 4$ there is a bijection

$$\underbrace{\left\{ \begin{array}{c} \text{smooth } n\text{-dim } M \text{ with stable} \\ \text{framed normal bundle} \end{array} \right\}}_{\text{cobordism}} \xrightarrow{\quad} \underbrace{\left\{ \begin{array}{c} \text{topological } n\text{-dim } M \text{ with stable} \\ \text{framed normal microbundle} \end{array} \right\}}_{\text{cobordism}}$$

Note that smoothing theory actually proves the stronger result that a topological manifold M with framed stable normal microbundle admits an essentially unique smooth structure with framed stable normal bundle. No cobordism is needed!

4.3. Outline of the proof. We will only give a proof in the case $n \geq 6$, for which we have established microbundle transversality, and the trivial sequential tangential structure $B\text{Top} = (B\text{Top}(n))_{n \geq 0}$ to avoid additional notation. It is essentially the same proof as in the smooth case, using the tools we have obtained in the previous lecture.

Our first goal is to describe maps

$$\begin{aligned} \mathcal{C}: \Omega_n^{\text{Top}} &\rightarrow \pi_n(M\text{Top}) \\ \mathcal{I}: \pi_n(M\text{Top}) &\rightarrow \Omega_n^{\text{Top}} \end{aligned}$$

4.3.1. The Pontryagin-Thom collapse map. The map \mathcal{C} is the so-called Pontryagin-Thom collapse map. We saw before that any n -dimensional compact topological manifold M admits a locally flat embedding into \mathbb{R}^s with a normal microbundle ν_{s-n} . Using the Kister-Mazur theorem, its total space $E \subset \mathbb{R}^s$ can be assumed to be the total space of an \mathbb{R}^{s-n} -bundle. The quotient $\mathbb{R}^s/(\mathbb{R}^s \setminus E)$ is naturally homeomorphic to $\text{Th}(\nu_{s-n})$.

We can classify ν_{s-n} by a map $f: M \rightarrow B\text{Top}(s-n)$, i.e. $f^*\gamma_{s-n} \cong \nu_{s-n}$. Thus we have an induced map $\text{Th}(\nu_{s-n}) \rightarrow \text{Th}(\gamma_{s-n})$ and by composition we obtain a map

$$c_s: S^s \rightarrow \mathbb{R}^s/(\mathbb{R}^s \setminus E) \cong \text{Th}(\nu_{s-n}) \rightarrow \text{Th}(\gamma_{s-n})$$

sending ∞ to ∞ .

If we take the embedding $M \hookrightarrow \mathbb{R}^s \hookrightarrow \mathbb{R} \times \mathbb{R}^s$, then we may take ν_{s+1-n} to be $\epsilon \oplus \nu_{s-n}$, classified by $M \rightarrow B\text{Top}(s-n) \rightarrow B\text{Top}(s+1-n)$, and use this data to construct c_{s+1} . Then the following diagram commutes

$$\begin{array}{ccc} S^1 \wedge S^s & \xrightarrow{S^1 \wedge c_s} & S^1 \wedge \text{Th}(\gamma_{s-n}) \\ \downarrow \cong & & \downarrow \\ S^{s+1} & \xrightarrow{c_{s+1}} & \text{Th}(\gamma_{s-n+1}) \end{array}$$

Thus taking its pointed homotopy class and letting $s \rightarrow \infty$, we get an element of $\pi_n(M\text{Top})$. This is $\mathcal{C}(M)$.

Lemma 4.18. $\mathcal{C}(M)$ is well-defined.

Sketch of proof. The choices we made were a locally flat embedding, a normal \mathbb{R}^{s-n} -bundle and a classifying map. We saw that any two locally flat embeddings are isotopic if we are allowed to increase the dimension arbitrarily. Similarly any two normal bundles can be assumed concordant and any two classifying maps are homotopic. We leave it to the reader to check that these isotopies, concordances and homotopies induce homotopies of Pontryagin-Thom collapse maps.

For the invariance under cobordism, we use a relative version of the Whitney embedding theorem to show that N may be embedded in $\mathbb{R}^s \times [0, \infty)$ so that ∂N is embedded in $\mathbb{R}^s \times \{0\}$ with normal microbundle and a normal microbundle of N in $\mathbb{R}^s \times [0, \infty)$ extending this (which exists by a relative version of Theorem 3.14). Then $\mathcal{C}(N)$ provides a null-homotopy for $\mathcal{C}(\partial N)$. \square

4.3.2. *The transverse inverse image map.* For the map \mathcal{I} , we start with an element $c \in \pi_n(M\text{Top})$. This is represented by a pointed homotopy class of maps $c_s: S^s \rightarrow \text{Th}(\gamma_{s-n})$. We note that $\text{Th}(\gamma_{s-n})$ contains the 0-section, which has an \mathbb{R}^{n-s} -bundle neighborhood. This means we can apply our microbundle transversality theorem 3.19 to make c_s microbundle transverse to this 0-section.

Remark 4.19. There is an alternative to using a generalization of Theorem 3.19 for X that are not manifolds. Since $B\text{Top}(s-n)$ has finitely generated homotopy groups, we may actually find manifolds $X_{s-n}(k)$ with k -connected maps $X_{s-n}(k) \rightarrow B\text{Top}(s-n)$, and one may use these instead.

In particular, this says that $(c_s)^{-1}(0\text{-section})$ is a compact topological submanifold of \mathbb{R}^s of codimension equal to the codimension of the 0-section, i.e. $s-n$. Thus we have obtained an n -dimensional compact topological manifold M , which we denote $\mathcal{I}(c)$.

Lemma 4.20. $\mathcal{I}(c)$ is well-defined.

Sketch of proof. We made a choice of s , representative $c_s: S^s \rightarrow \text{Th}(\gamma_{s-n})$ and a transverse perturbation of this map. We may assume that $s' = s$ by increasing one of them, and then any two transverse perturbations c_s and c'_s are homotopic by a homotopy h . Then we can apply a relative version of the transversality result to the homotopy h to obtain a map $h: S^s \times [0, 1] \rightarrow \text{Th}(\gamma_{s-n})$ transverse to the 0-section and equal to c_s, c'_s at 0, 1 respectively. The inverse image of the 0-section under h gives a cobordism $\mathcal{I}(h)$ between $\mathcal{I}(c_s)$ and $\mathcal{I}(c'_s)$. This argument also shows that $\mathcal{I}(c)$ only depends on the stable homotopy class of c . \square

4.3.3. *\mathcal{C} and \mathcal{I} are mutually inverse.* There are two compositions to consider;

$$\mathcal{C} \circ \mathcal{I}: \pi_n(M\text{Top}) \rightarrow \pi_n(M\text{Top}) \quad \text{and} \quad \mathcal{I} \circ \mathcal{C}: \Omega_n^{\text{Top}} \rightarrow \Omega_n^{\text{Top}}.$$

Lemma 4.21. We have that $\mathcal{I} \circ \mathcal{C} = \text{id}$.

Proof. If we pick the representative $S^s \rightarrow \text{Th}(\gamma_{s-n})$ appearing in the construction of the stable map $\mathcal{C}(M)$, then we see that it already is microbundle transverse to the 0-section. The inverse image is exactly M . \square

Lemma 4.22. We have that $\mathcal{C} \circ \mathcal{I} = \text{id}$.

Sketch of proof. The inverse image $\mathcal{I}(c_s)$ is an n -dimensional topological manifold M with an embedding into \mathbb{R}^s and a normal microbundle ν_{s-n} , by definition of microbundle transversality. The map c_s restricts to a map from M to $B\text{Top}(s-n)$ and embeds each fiber of ν_{s-n} into a fiber of γ_{s-n} .

If we shrink ν_{s-n} , we may assume that the image of c_s is exactly the ϵ -bundle in γ_{s-n} for some small $\epsilon > 0$. We can then reparametrize the fibers of ν_{s-n} and scale the image of c_s , so that we can assume ν_{s-n} is an \mathbb{R}^{s-n} -bundle and we have a map $c'_s: \nu_{s-n} \rightarrow \gamma_{s-n}$ of \mathbb{R}^{s-n} -bundles. We claim that if we use c'_s to construct a map $\tilde{c}_s: S^s \rightarrow \text{Th}(\gamma_{s-n})$, this is homotopic to c_s preserving the base point ∞ . To see that this is the case, note that \tilde{c}_s differs from c_s by collapsing a larger neighborhood of ∞ and reparametrizing. This may be seen to be homotopic to c_s with some effort. \square

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E-mail address: kupers@math.ku.dk

INSTITUT FOR MATEMATISKE FAG, KØBENHAVNS UNIVERSITET, UNIVERSITETSPARKEN 5, 2100 KØBENHAVN Ø.