Iterated Monodromy for a Two-Dimensional Map

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Abstract. We compute the iterated monodromy group for a postcritically finite endomorphism F of $\mathbb{P}^2$. The postcritical set is the union of six lines, and the wreath recursion for the group closely reflects the dynamics of F on these lines.

Introduction

In [**BN**], L. Bartholdi and V. Nekrashevych solved the *twisted rabbit problem* with iterated monodromy groups. Their work has brought new tools to bear in the fields of dynamics and algebra. In [**N2**], V. Nekrashevych uses a more general notion of iterated monodromy group to obtain combinatorial models for Julia sets of certain maps of several complex variables. Other than this, little has been done with iterated monodromy groups in dimensions greater than one. Here we compute the iterated monodromy group for a postcritically finite endomorphism $F : \mathbb{P}^2 \to \mathbb{P}^2$. The ideas used in this computation could generalize to calculate the iterated monodromy groups for other maps $\mathbb{P}^n \to \mathbb{P}^n$.

Let $F \colon \mathbb{C}^2 \to \mathbb{C}^2$ be the following rational function:

$$F(x, y) = \left(1 - \frac{y^2}{x^2}\ ,\ 1 - \frac{1}{x^2}\right).$$

Then F extends to a holomorphic endomorphism of the complex projective plane $\mathbb{P}^2$, i.e. an everywhere-defined holomorphic map $\mathbb{P}^2 \to \mathbb{P}^2$. In homogeneous coordinates, this endomorphism is given by $F(x : y : z) = (x^2 - y^2 : x^2 - z^2 : x^2)$.

Topologically, the map F is a branched cover of degree four, with fibers of the form $\{(x, y), (-x, y), (x, -y), (-x, -y)\}$. The critical locus of F is the union of the complex lines $x = 0$ and $y = 0$ in $\mathbb{C}^2$, as well as the line at infinity $L_\infty := \mathbb{P}^2 \setminus \mathbb{C}^2$, and F restricts to a covering map on the complement of these lines.

The *postcritical locus* of F is the forward orbit of the critical locus. A map is called *postcritically finite* if the postcritical locus is an algebraic set, i.e. the union of finitely many algebraic varieties. (Postcritically finite endomorphisms were first studied by Fornæss and Sibony in [**FS**].) Our map F is postcritically finite, and the postcritical locus is the union of six lines:

$$\Delta \;=\; \{x = 0\} \cup \{y = 0\} \cup L_\infty \cup \{x = 1\} \cup \{y = 1\} \cup \{y = x\}.$$

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The map F permutes these lines as follows:

$$\begin{tikzcd} x=0 \arrow[r] & L_\infty \arrow[d] \\ y=x \arrow[u] & y=1 \arrow[l] \end{tikzcd} \qquad\qquad \begin{tikzcd} y=0 \arrow[d, bend left] \\ x=1 \arrow[u, bend left] \end{tikzcd}$$

Any postcritically finite endomorphism restricts to a partially-defined covering map on the complement of the postcritical locus. That is, F restricts to a covering map $F\colon X_1 \to X$, where $X = \mathbb{P}^2 \setminus \Delta$ and $X_1 \subset X$. This partial self-cover has an associated iterated monodromy group, which describes the topology of the cover and can be used to create combinatorial models of the associated Julia set.

Here is our main result:

Theorem. *The iterated monodromy group for the map F can be defined by the following wreath recursion:*

$$\begin{aligned}
a &= \langle b, 1, 1, b\rangle \\
b &= \langle c, c, 1, 1\rangle \\
c &= \langle d, d_y, d_x, 1\rangle\,(1\;4)(2\;3) \\
d &= \langle 1, a, 1, a\rangle\,(1\;2)(3\;4) \\
e &= \langle f, 1, f, 1\rangle \\
f &= \langle b^{-1}, 1, be, e\rangle\,(1\;3)(2\;4)
\end{aligned}$$

where $d_x = (af)^{-1}$ and $d_y = (bec)^{-1}$.

In section 1 we give the necessary background on tree automorphisms, wreath recursions, and iterated monodromy groups, and section 2 is devoted to a proof of this theorem.

The map F is very special; it arises naturally as a map on a certain *moduli space*, and there is a certain amount of Teichmüller theory underlying the construction. This provides a link between the dynamics of this map $F : \mathbb{P}^2 \to \mathbb{P}^2$, and the dynamics of a particular holomorphic map, the *Thurston pullback map*, on an appropriate *Teichmüller space* (see [**DH**]). The thesis [**K**] contains the details of this calculation.

Ordinarily, complex dynamics in several variables is quite difficult; few of the techniques from one variable dynamics carry over to higher dimensions. As demonstrated in [**N1**], the techniques of iterated monodromy groups can be used to obtain combinatorial models for Julia sets; understanding the structure of the Julia set is a key part of understanding the dynamics.

1. Background

1.1. Tree Automorphisms. Let T_d be the tree of all finite d-ary sequences. For example, the tree T_2 of finite binary sequences is shown in figure 1. The vertices of T_d are finite sequences of digits $\{1, \ldots, d\}$, and the edges are pairs of the form $\{\omega, k\omega\}$, where ω is a finite d-ary sequence and $k \in \{1, \ldots, d\}$.

An *automorphism* of T_d is a bijection of the vertices that maps edges to edges. By convention, automorphisms of T_d will act on the *right*. That is, $\omega \cdot \alpha$ will denote the automorphism α applied to the d-ary sequence ω. Similarly, the composition $\alpha\beta$ of two automorphisms will denote α followed by β, that is $\omega \cdot (\alpha\beta) = (\omega \cdot \alpha) \cdot \beta$.

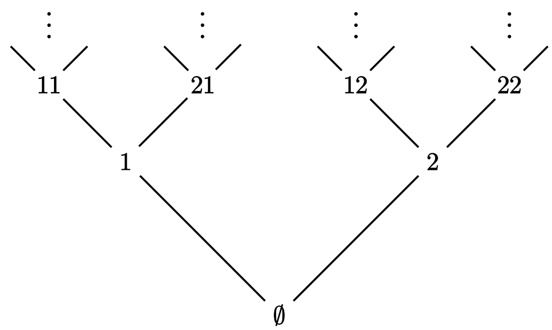

FIGURE 1. The infinite binary tree T_2.

There are two basic kinds of automorphisms of T_d:

(1) If σ is a permutation of the set $\{1, \ldots, d\}$ (acting on the right), then σ can be extended to an automorphism of T_d by the rule

$$(\omega k) \cdot \sigma = \omega(k \cdot \sigma).$$

That is, σ simply permutes the first-level subtrees of T_d.

(2) If $\alpha_1, \ldots, \alpha_d$ are automorphisms of T_d, we can define an automorphism $\langle \alpha_1 \ldots \alpha_d \rangle$ of T_d by the rule

$$(\omega k) \cdot \langle \alpha_1 \ldots \alpha_d \rangle = (\omega \cdot \alpha_k) k$$

That is, $\langle \alpha_1 \ldots \alpha_d \rangle$ acts trivially on the first-level vertices of T_d, and restricts to the automorphisms $\alpha_1, \ldots, \alpha_d$ on the first-level subtrees.

Any automorphism α of T_d can be written uniquely as a product

$$\alpha = \langle \alpha_1, \ldots, \alpha_d \rangle \, \sigma$$

where $\alpha_1, \ldots, \alpha_d$ are automorphisms of T_d and σ is a permutation of $\{1, \ldots, d\}$.

The discussion above amounts to a structure theorem for the automorphism group $\mathrm{Aut}(T_d)$. Specifically, $\mathrm{Aut}(T_d)$ can be written as a semidirect product

$$\mathrm{Aut}(T_d) \cong \mathrm{Aut}(T_d)^d \rtimes \Sigma_d$$

where Σ_d denotes the permutation group on the set $\{1, \ldots, d\}$, and Σ_d acts on $\mathrm{Aut}(T_d)^d$ by permutation of factors:

$$\langle \alpha_1, \ldots, \alpha_d \rangle \, \sigma \, \langle \beta_1, \ldots, \beta_d \rangle \, \tau \;=\; \langle \alpha_1 \beta_{1 \cdot \sigma}, \ldots, \alpha_d \beta_{d \cdot \sigma} \rangle \, \sigma\tau.$$

A semidirect product of the form $G^n \rtimes \Sigma_n$ with Σ_n acting by permutation of factors is known as a *wreath product*, and is usually denoted $G \wr \Sigma_n$. The results above are summarized by the following theorem which can be found in [**N1**]:

THEOREM 1.1. *Let T_d be the infinite d-ary tree, and let Σ_d denote the permutation group on the set $\{1, \ldots, d\}$. Then:*

$$\mathrm{Aut}(T_d) \;\cong\; \mathrm{Aut}(T_d) \wr \Sigma_d.$$

That is, $\mathrm{Aut}(T_d)$ *is isomorphic to the infinite wreath product* $((\cdots \wr \Sigma_d) \wr \Sigma_d) \wr \Sigma_d$.

We can use this description of $\mathrm{Aut}(T_d)$ to define automorphisms recursively. For example, consider the following equation:

$$\alpha = \langle 1, \alpha \rangle \, (1\ 2)$$

This equation describes an automorphism $\alpha \in \mathrm{Aut}(T_2)$ with the following properties:

(1) The automorphism α swaps the first-level subtrees of T_2.
(2) Neglecting this swap, α acts trivially on the left subtree, but acts as α on the right subtree.

These conditions uniquely determine an automorphism $\alpha \in \mathrm{Aut}(T_2)$.

More generally, a set of automorphisms can be defined using a recursive system of equations. For example, the Grigorchuk group of intermediate growth is the subgroup of $\mathrm{Aut}(T_2)$ generated by elements $\alpha, \beta, \gamma, \delta$ defined by the following equations:

$$\alpha = (1\ 2) \qquad \beta = \langle \alpha, \gamma \rangle \qquad \gamma = \langle \alpha, \delta \rangle \qquad \delta = \langle 1, \beta \rangle.$$

See [**dlH**] for more information on wreath products, automorphisms of trees, and the Grigorchuk group.

A subgroup $G \leq \mathrm{Aut}(T_d)$ is called *self-similar* if, for every automorphism $\langle \alpha_1, \ldots, \alpha_d \rangle \sigma \in G$, each automorphism α_k also lies in G. Equivalently, G is self-similar if the isomorphism $\mathrm{Aut}(T_d) \to \mathrm{Aut}(T_d) \wr \Sigma_d$ restricts to an inclusion of G into $G \wr \Sigma_d$. Any finitely-generated self-similar group can be specified via a recursive system of equations for the generators:

$$\begin{aligned} g_1 &= \langle g_{11}, \ldots, g_{1d} \rangle\, \sigma_1 \\ &\ \vdots \\ g_n &= \langle g_{n1}, \ldots, g_{nd} \rangle\, \sigma_n \end{aligned}$$

Here each g_{ij} is a product of the generators $g_1, \ldots, g_n$ and their inverses. A system of equations of this form is known as a *wreath recursion*.

1.2. Iterated Monodromy Groups. Let X be a topological space. A *partial self-covering* of X is a covering map $f\colon X_1 \to X$, where X_1 is an open subset of X. For example, if $f\colon \mathbb{P}^n \to \mathbb{P}^n$ is a postcritically finite endomorphism with postcritical locus Δ, then f restricts to a partial self-covering of $\mathbb{P}^n \backslash \Delta$, with domain $\mathbb{P}^n \backslash f^{-1}(\Delta)$.

If we iterate a partial self-covering $f\colon X_1 \to X$, we obtain maps

$$f^n\colon X_n \to X.$$

where $X_n = f^{-n}(X)$ is the domain on which f^n is defined. If f has degree d, then f^n is a partial self-covering of X with degree d^n.

Choose a basepoint $t \in X$. The *backwards orbit* of t is the disjoint union

$$T = \coprod_{n \geq 0} f^{-n}(t).$$

The backwards orbit T has the structure of an infinite d-ary tree, with edges corresponding to the action of f. The root of this tree is the basepoint t, the first-level vertices are the d elements of $f^{-1}(t)$, the second-level vertices are the d^2 elements of $f^{-2}(d)$, and so forth.

The fundamental group $\pi_1(X, t)$ acts on the tree T by monodromy. Specifically, if α is an oriented loop in X based at t, then α lifts to one oriented path starting at each vertex v of T, and we define $v \cdot \alpha$ to be the endpoint of this path. This action defines a homomorphism

$$\pi_1(X, t) \to \mathrm{Aut}(T)$$

where $\mathrm{Aut}(T)$ is the automorphism group of T. The image of this homomorphism is the *iterated monodromy group* of f based at t, denoted $\mathrm{IMG}(f, t)$. Equivalently,

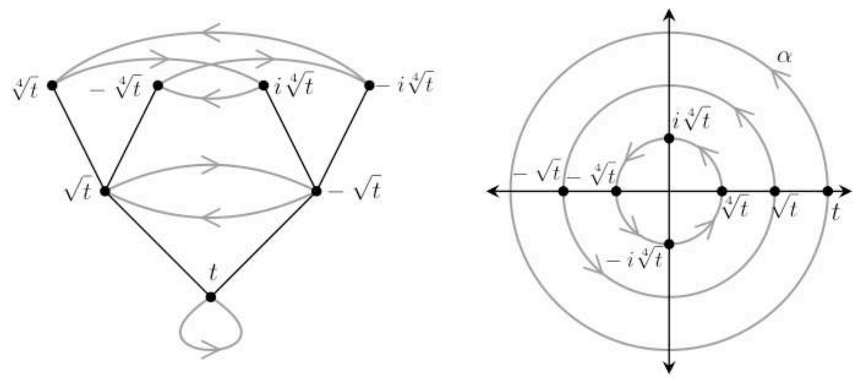

FIGURE 2. The iterated monodromy action for the map $f(z) = z^2$.

the iterated monodromy group can be defined as the quotient of the fundamental group $\pi_1(X, t)$ by the kernel of the monodromy action on T. We will often blur the distinction between a loop α in X based at t, the corresponding group element $\alpha \in \pi_1(X, t)$, and the resulting automorphism α of the tree T.

EXAMPLE 1.2. Figure 2 shows the iterated monodromy action for the map $f(z) = z^2$, where $X = \mathbb{C} \setminus \{0\}$. The basepoint is chosen to be a real number $t > 1$, and only the first few lifts of the basepoint are shown. The loop α represents a generator of $\pi_1(X, t)$. Its preimage $f^{-1}(\alpha)$ is the union of a path from $\sqrt{t}$ to $-\sqrt{t}$ and a path from $-\sqrt{t}$ to $\sqrt{t}$, so:

$$\sqrt{t} \cdot \alpha = -\sqrt{t} \qquad \text{and} \qquad \left(-\sqrt{t}\right) \cdot \alpha = \sqrt{t}$$

Similarly, the second preimage $f^{-2}(\alpha)$ consists of four paths, each starting at a vertex v and ending at iv. The resulting action is shown in the figure. In this case, the action of $\pi_1(X, t)$ on the tree T is faithful (with α^n acting nontrivially on the nth level of the tree), so the iterated monodromy group is isomorphic to $\pi_1(X, t) \cong \mathbb{Z}$. □

If we wish to analyze the structure of $\mathrm{IMG}(f, t)$, we must define an isomorphism $T_d \to T$, where T_d is the tree of finite d-ary sequences. To that end, choose *connecting paths* $\ell_1, \ldots, \ell_d$ in X from the basepoint t to each of the first-level vertices in T. Lifting these paths under the iterates of f, we obtain one lift of ℓ_k starting at v for each $k \in \{1, \ldots, d\}$ and each vertex $v \in T$. This allows us to define an isomorphism $\omega \to t_\omega$ inductively by the following rule:

The vertex $t_{\omega k}$ is the endpoint of the lift of ℓ_k starting at t_ω.

The base case is $t_\emptyset = t$, which makes $t_1, \ldots, t_d$ the endpoints of the paths $\ell_1, \ldots, \ell_d$, respectively.

NOTE 1.3. Observe that the rule for the isomorphism $T_d \to T$ involves appending digits to the *right* of a d-ary sequence, while the rule for adjacency in T_d involves appending digits to the *left*. This is because the lifts of the connecting paths $\ell_1, \ldots, \ell_k$ do *not* correspond to edges in T. Instead, the lifts of ℓ_k connect each vertex v to $\delta_k(v)$, where δ_k is an isomorphism between T and one of its first-level subtrees.

Identifying T and T_d, we can now regard any loop α in X based at t as an automorphism of the infinite d-ary tree T_d. The following proposition found in [**N1**] explains how to calculate this automorphism:

Proposition 1.4. *Let α be a loop in X based at t, and let $\langle \alpha_1, \ldots, \alpha_k \rangle\, \sigma$ be the associated automorphism of T_d. Then the permutation σ is determined by the monodromy action of α:*

$$t_k \cdot \alpha \;=\; t_{k\cdot\sigma}$$

Furthermore, α_k is the automorphism of T_d associated to the loop

$$\ell_k \cdot \tilde{\alpha} \cdot \ell_{k\cdot\sigma}^{-1}$$

where $\tilde{\alpha}$ is the lift of α starting at t_k. □

It follows immediately from this proposition that the iterated monodromy group $\mathrm{IMG}(f,t)$ is self-similar when regarded as a subgroup of $\mathrm{Aut}(T_d)$.

2. The Set-Up

Recall the map $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$F(x,y) = \left(1 - \frac{y^2}{x^2},\, 1 - \frac{1}{x^2}\right).$$

As discussed in the introduction, this map restricts to a partially-defined covering map $F\colon X_1 \to X$, where

$$X \;=\; \mathbb{C}^2 \setminus \big(\{x=0\} \cup \{y=0\} \cup \{x=1\} \cup \{y=1\} \cup \{y=x\}\big)$$

and $X_1 = F^{-1}(X) = X \setminus \big(\{x=-1\} \cup \{y=-1\} \cup \{y=-x\}\big)$.

Paths in X. For purposes of visualization, we can regard $\mathbb{C}^2$ as the configuration space of two points x, y on the complex plane. From this point of view, X is the configuration space of two distinct points $x \neq y$ in the twice-punctured plane $\mathbb{C} \setminus \{0,1\}$.

A path in X is a pair (p_x, p_y), where p_x and p_y are paths in $\mathbb{C}\setminus\{0,1\}$ describing the motions of the points x and y, respectively. A path for which y is fixed is called an *x-path*, and a path for which x is fixed is called a *y-path*.

Conventions 2.1. We shall use the following conventions for figures:

(1) Positions for x will be drawn as closed dots, and positions for y will be drawn as open circles.
(2) Paths for x will be drawn as solid lines, and paths for y will be drawn as dotted lines.
(3) The points 0 and 1 will be marked by crosses ($\times$).

Basepoint and Connecting Paths. We must choose a basepoint for X. For convenience, we shall use one of the fixed points of F, namely the point

$$t \;=\; (x_0, y_0) \;\approx\; (0.66 + 1.11i,\, 1.28 + 0.53i).$$

This point has four preimages:

$$t_1 = t = (x_0, y_0) \qquad t_2 = (-x_0, y_0) \qquad t_3 = (x_0, -y_0) \qquad t_4 = (-x_0, -y_0)$$

Figure 3 shows each of these four points, as well as connecting paths $\ell_1, \ell_2, \ell_3, \ell_4$ from these points to the basepoint t. (Since $t_1 = t$, the path ℓ_1 is trivial.)

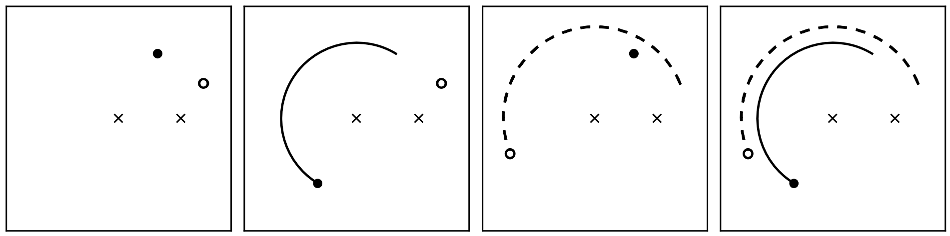

FIGURE 3. The connecting paths ℓ_1, ℓ_2, ℓ_3, and ℓ_4.

Generating Loops. The fundamental group $\pi_1(X, t)$ is generated[1] by the six loops a, b, c, d, e, f shown in figure 4. Each of these generators encircles one of the six lines of Δ, as shown in the following table:

Generator	a	b	c	d	e	f
Line	$x = 0$	$y = x$	$y = 1$	L_∞	$y = 0$	$x = 1$

The lifts of these generators based at the basepoint t exactly mimic the action of F on the six lines of Δ. In particular:

(1) The lift of a based at t is homotopic to b,
(2) The lift of b based at t is homotopic to c,
(3) The lift of c^2 based at t is homotopic to d,
(4) The lift of d^2 based at t is homotopic to a,
(5) The lift of e based at t is homotopic to f, and
(6) The lift of f^2 based at t is homotopic to e.

As you can see in figure 4, the loops a and f only involve motion of the point x, while the loops c and e only involve motion of the point y. As we have drawn it,

[1] The fundamental group of the complement of a hyperplane arrangement in $\mathbb{C}^n$ is well-understood. In particular, there always exists a generating set consisting of one loop around each hyperplane. See [**OT**].

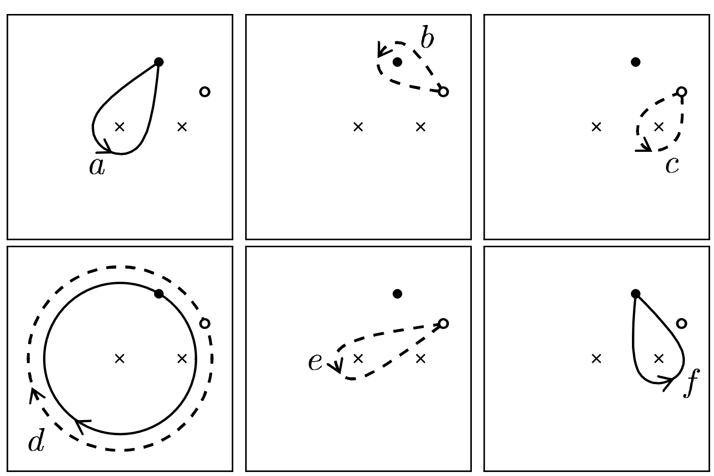

FIGURE 4. The six generators of $\pi_1(X, t)$.

the loop b involves the point x moving counterclockwise around y, but b is also homotopic to a loop where y moves counterclockwise around x.

Finally, the loop d involves motion of both x and y. In particular, we can write d as a product:

$$d = d_x d_y$$

where $d_x \in \pi_1(X, t)$ only involves motion of x, and $d_y \in \pi_1(X, t)$ only involves motion of y. Because the x and y paths are disjoint, the loops d_x and d_y commute up to homotopy. Moreover, each of these loops can be expressed in terms of the other generators:

$$d_x = f^{-1}a^{-1} \qquad \text{and} \qquad d_y = e^{-1}c^{-1}b^{-1}.$$

3. The Computation

We are now in a position to calculate the wreath recursion for $\mathrm{IMG}(F, t)$. According to proposition 1.4, the recursive equation for a loop $\alpha \in \pi_1(X, t)$ is

$$\alpha \;=\; \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \, \sigma$$

where σ is a permutation of $\{1, 2, 3, 4\}$ representing the mondromy action of α on $\{t_1, t_2, t_3, t_4\}$, and

$$\alpha_i \;=\; \ell_i \cdot \tilde{\alpha}_i \cdot \ell_{i\cdot\sigma}^{-1}.$$

Here ℓ_i is the connecting path to t_i, $\tilde{\alpha}_i$ is the lift of α beginning at t_i, and $\ell_{i\cdot\sigma}$ is the connecting path to the endpoint of $\tilde{\alpha}_i$.

Theorem 3.1. *Using the basepoint t, the connecting paths $\ell_1, \ell_2, \ell_3, \ell, 4$, and the generators a, b, c, d, e, f, the wreath recursion for* $\mathrm{IMG}(F)$ *is the following:*

$$\begin{aligned}
a &= \langle b, 1, 1, b \rangle \\
b &= \langle c, c, 1, 1 \rangle \\
c &= \langle d, d_y, d_x, 1 \rangle \, (1\ 4)(2\ 3) \\
d &= \langle 1, a, 1, a \rangle \, (1\ 2)(3\ 4) \\
e &= \langle f, 1, f, 1 \rangle \\
f &= \langle b^{-1}, 1, be, e \rangle \, (1\ 3)(2\ 4)
\end{aligned}$$

where $d_x = (af)^{-1}$ and $d_y = (bec)^{-1}$.

Proof. The lifts of the six generators are shown in figure 5. We will explain each part of the wreath recursion in turn.

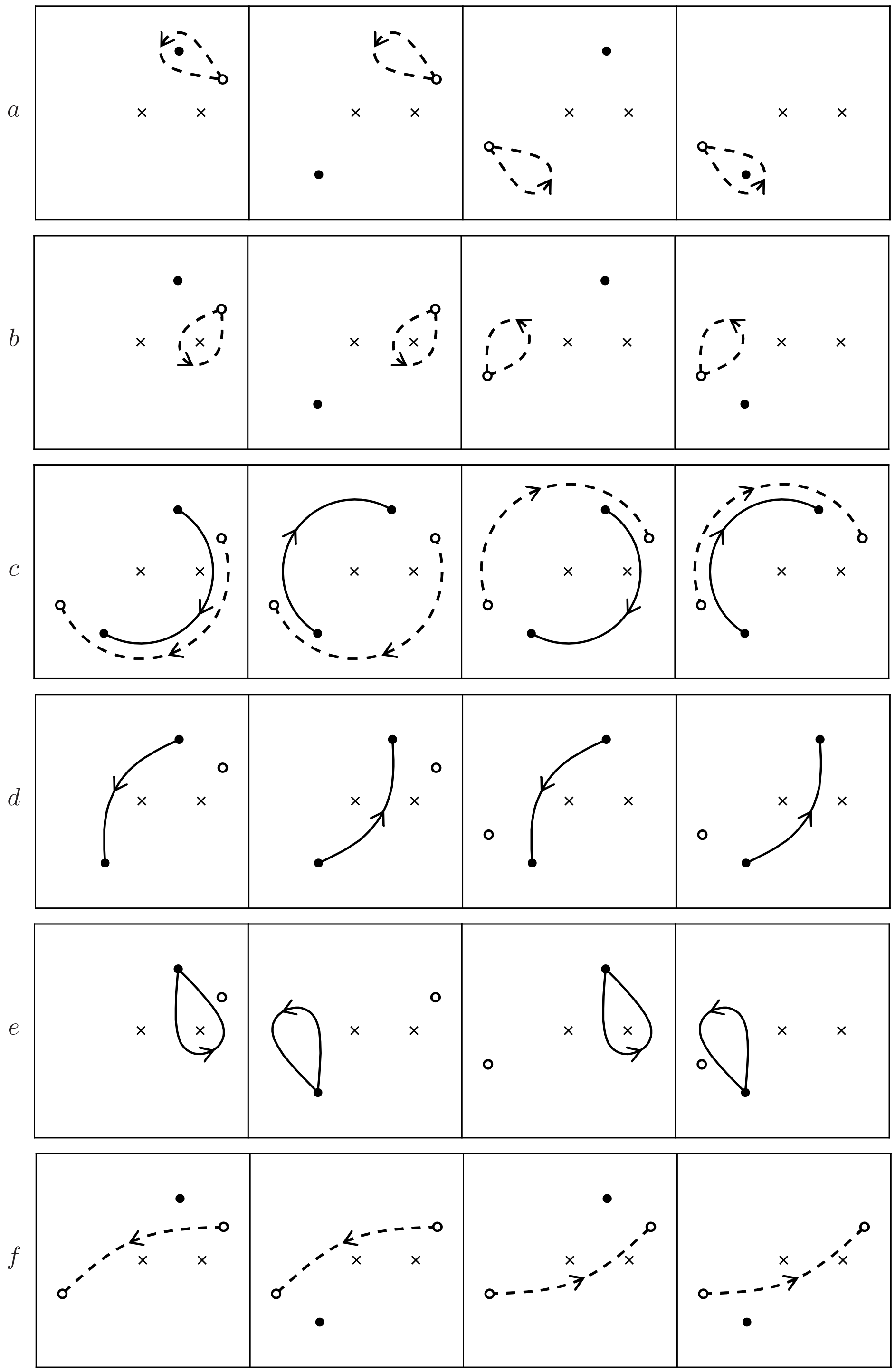

FIGURE 5. The lifts of the generators.

Lifting generator a. The path a_1 is homotopic to b, and a_2 is trivial since the x-loop around y_0 is nullhomotopic. The path a_3 is trivial since the x-loop around $-y_0$ is nullhomotopic. Finally, the path a_4 is homotopic to b. In particular, a_4 consists of the path ℓ_4, followed by a loop in which y moves counterclockwise around x, followed by the path ℓ_4^{-1}. By contracting the beginning and ending paths, the counterclockwise motion of y around x can be moved through the third and second quadrants to the first quadrant, resulting in the loop b. Since the lifts of a are all loops, the permutation associated with a is trivial.

Lifting generator b. The path b_1 is homotopic to c, and the path b_2 is homotopic to c since the motion of the point x (from x_0 to $-x_0$ along ℓ_2, and then back along ℓ_2^{-1}) is homotopically trivial. The paths b_3 and b_4 are trivial, since the y-loop around -1 is nullhomotopic. Since the lifts of b are all loops, the permutation associated with b is trivial.

Lifting generator c. Path c_1 is homotopic to d since the lift of c moves x and y half a rotation clockwise, and the connecting path ℓ_4^{-1} completes the clockwise rotation. The path c_2 is homotopic to d_y. The initial connecting path ℓ_2 moves x counterclockwise, and the lift of c moves it clockwise, resulting in a trivial motion of x. However, the point y move clockwise under the lift of c, and then moves clockwise again during the final connecting path ℓ_3^{-1}, resulting in a full clockwise rotation for y.

The path c_3 is homotopic to d_x. The point x moves clockwise under the lift of c and clockwise again under ℓ_2^{-1}, while y moves counterclockwise under ℓ_3 and then clockwise under the lift of c. And finally the path c_4 is trivial since both x and y move counterclockwise under ℓ_4, and then clockwise under the lift of c. The lifts of c are paths connecting t_1 with t_4 and t_2 with t_3, so the permutation associated with c is $(1\,4)(2\,3)$.

Lifting generator d. The path d_1 is trivial since the lift of d moves x counterclockwise around 0, and then the connecting path ℓ_2^{-1} moves x clockwise. The path d_2 is homotopic to a as the connecting path ℓ_2 moves x counterclockwise around 0, and then the lift of d continues the counterclockwise motion, resulting in a complete loop around 0. The path d_3 is trivial, for the same reason as d_1. The path d_4 is homotopic to a, for the same reason as d_2; the motion of y from y_0 to $-y_0$ and back to y_0 is homotopically trivial. The lifts of d are paths connecting t_1 with t_2 and t_r with t_3, so the permutation associated with d is $(1\,2)(3\,4)$.

Lifting generator e. The paths e_1 and e_3 is homotopic to f. The paths e_2 and e_4 are trivial, since the x-loop around -1 is nullhomotopic. Since the lifts of e are all loops, the permutation associated with b is trivial.

Lifting generator f. The path f_1 is homotopic to b^{-1}; the point y moves along the indicated path under the lift of f, and then moves clockwise along the outside under the connecting path ℓ_3^{-1}. The result is that y moves clockwise around x. The path f_2 is trivial; first x moves down under ℓ_2, then y moves left under the lift of f, then y moves right and x moves back up under ℓ_4^{-1}.

The path f_3 is homotopic to be. First y moves counterclockwise around the outside under ℓ_3, and then y moves along the indicated path back to y_0 under the lift of f. The result is that y moves counterclockwise around both x and 0, which is homotopic to the loop b followed by the loop e.

The path f_4 is homotopic to e, though this is not quite obvious. The problem is that the paths traveled by x and y cross, making it slightly difficult to disentangle

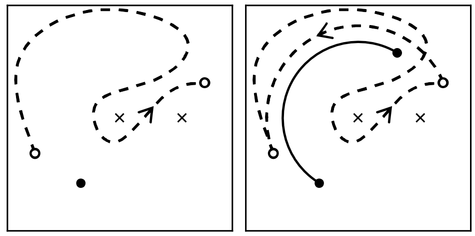

FIGURE 6. By redrawing the lift of f, we can make the x and y paths of f_4 disjoint.

the motion. The solution is to redraw the lift of f as in figure 6, so that the path of y goes around the outside of x_0. This makes the paths of x and y disjoint, with x moving trivially and y moving counterclockwise around 0.

Finally, the lifts of f are paths connecting t_1 with t_3 and t_2 with t_4, so the permutation associated with f is $(1\ 3)(2\ 4)$.

□

The iterated monodromy group for F is actually *contracting*; this is because the map F is expanding on its Julia set. Using Mathematica, we were able to calculate the *nucleus* for this recursion. According to our program, the nucleus has 59 elements, consisting of the identity element plus the following 29 elements and their inverses:

$$a, b, c, d, e, f, ab, ac, af, ac^{-1}, bc, bd, be, bd^{-1}, cd, cf, da,$$
$$de, ea, ec, fb, fd, abe, acf, afb, afd, bcf, eac, f^{-1}be.$$

See [**BN**] or [**N1**] to learn more about contracting actions and the nucleus associated with a recursion.

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