Quantum Multivariable Calculus

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0.1 Introduction

Calculus is any theory in which there is a differentiation $f \to f'$ and integration $f \to \int f$ operation in such a way that $(\int f)' = f$. This does not necessarily mean that the notion of derivative or integral are the classical notions.

A **quantum calculus** is a version of calculus in which we do not take limits. Derivatives are differences and anti derivatives are sums. It is a theory, where smoothness is no more required. With a suitable generalization of integration of differential forms on curves and surfaces, most notions of multi-variable calculus generalize like Stokes theorem. Actually this extension makes many proofs simpler. There is a larger symmetry in the theory because "curves" are on the same footing than 1-forms, the space of surfaces is the same than than the space 2-forms, points with functions and solids with volume forms. The process of taking limits is involved when deriving classical calculus from it and for some notions like local extrema. I personally find it important that any quantum calculus theory should not be an alternative to classical calculus but an **extension** of classical calculus but that it should stand on its own feet if possible. Any result obtained in quantum calculus become in the classical limit results in classical calculus. This limit has the handicap that it can only be taken for special objects like smooth functions. The more general theory works with continuous functions or even with their completion in a Hilbert space.

The specification "quantum" is used in many different contexts. Mathematical objects can be "quantized" in different ways. One can discretize, replace commutative algebras with non-commutative algebras, or replace classical variational problems with path integrals. We will look here at "quantum calculus" in the sense of Kac and not "quantized calculus" as introduced by Connes. One aspect of the later is part of a more **general quantum calculus** where one takes a measure μ on the real line and where

$$f' = \int (f(x+h) - f(x))/h \; d\mu(h) \; .$$

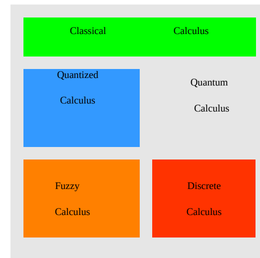
Quantized calculus is the special case, when μ is the Lebesgue measure and **quantum calculus** is when μ is a Dirac measure:

$$f' = \frac{f(x+h) - f(x)}{h} \; .$$

The measure μ which defines the derivative has a smoothing effect if it is applied to space of derivatives. It appears that quantum calculus is a convenient choice. With a more general measure, it is for example more difficult to deal with partial differential equations.

For Connes, quantized calculus is given by a derivative $df = [F, f]$, where $F : H \to H$ is a selfadjoint operator which satisfies $F^2 = 1$. Real variables are replaced by self-adjoint operator, infinitesimals become compact operators. The integral is the Dixmier trace. Measurable operators are ones for which $(1/log(N)) \sum_{n \leq N} \mu_n$ converges along a subsequence. The Dixmier trace neglects infinitesimals of the order > 1 operators for which $\mu_n = o(n^{-1})$. Compare the zeta function $tr(T^s)$. A Fredholm module is a representation of an algebra A as operators in H such that $[F, f]$ is infinitesimal (compact) for all f in A. For single variable calculus, F is the Hilbert transform.

Classical mechanics can be formalized with more general differential structures. Similarly, one can chose the Planck constant $h = h(x)$ to be space dependent. Such a generalization help to formulate part of the theory and Stokes theorem survives in the same way as it extends to Riemannian geometry in the classical case.



Quantum calculus includes classical calculus as a limit. There are various special cases, like discrete calculus or quantized calculus. Quantum calculus is formally easier than classical calculus. With suitable notation, it can be taught in the same way as classical calculus.

What are the advantages of quantum calculus?

- The formalism is simple. Many technicalities disappear.
- It allows the numerical computations in calculus like fluxes, line integrals without having to parameterize surfaces or curves.
- Stokes theorem has an simpler form. The difficulty of the notation disappears.
- Many proofs become elegant.
- Differential forms and geometric objects are treated in the same way.
- It gives deformations of classical notions and allows to reflect better what notions of mathematics are natural.
- It allows to do calculus on continuous functions which do not need to be smooth.

What are the disadvantages of quantum calculus?

- Not everything generalizes. Notably the chain rule needs adaptation.
- The discretisation breaks symmetries like rotational symmetries.
- Integrable differential equations become non-integrable after the discretization.
- We need to deform well established notions to keep classical things true.
- The topic is not taught yet in intro classes.
- Basic problems as Kepler's problem do not have simple enough quantum versions yet.

Is quantum calculus relevant? Things are relevant if they are useful and allow to solve problems. This usefulness test definitely applies to classical calculus. Because quantum calculus with fixed h is just calculus without limits, it can be considered a generalization of classical calculus. It inherits the relevance of classical calculus.

Can one imagine an other intelligence somewhere in the universe which use discrete calculus instead of quantum calculus? Can they solve concrete problems with it, like predict the motion of stars and planets, unlock the secrets of nature like particle physics without actually doing any limit? We might never know, but I believe that we can explore the usefulness of the question here on earth.

Can one recover classical calculus from quantum calculus? There is the limit $h \to 0$. But there are other connections. The **sampling theorem** in Fourier theory assures that for a band-limited function f, a function for which $\hat{f}(\omega) = 0$ for $|\omega| > h$ the **Nyquist frequency**, one can recover the function from the samples $f_n = f(n\pi/h)$ (see [**?**] appendix):

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \text{sinc}(ht - n\pi) = g(t) ,$$

where $\text{sinc}(x) = \sin(x)/x$. This implies that for $h = \pi$ we have a relation between the classical derivative and the values of f on the grid points:

$$f'(0) = \sum_{n \neq 0} (-1)^{n+1} f_n/n .$$

What is the relation with non-standard calculus?
Quantum calculus takes a discrete derivative with a constant h. If this constant h is infinitesimal, then it becomes usual calculus if the objects allow the transition to infinitesimal h. While **nonstandard calculus** is an elegant way to treat calculus, it has disadvantages. There are justifications needed which are not so easy to teach without a solid logic background. Nelsons version is better than Robinsons in this respect but still, some logic background is needed. In my experience, there are also risks of making mistakes in nonstandard analysis. Both problems could be overcome but it is unlikely to happen soon. The teaching of calculus never change radically. In our experience, working with infinitesimal h is as difficult as taking limits in the traditional setup. Quantum calculus does not do the limit $h \to 0$ nor assume h to be infinitesimal. It remains a positive real number.

Quantum calculus is constantly used. In solid state physics for example, there is the **tight binding approximation** which replaces derivative with finite differences. Often, objects are modeled directly in a discrete form, using difference equations. Finally, numerical computations of classical calculus objects are often discrete versions. In some sense, the computer computes with quantum calculus.

Can quantum calculus replace classical calculus? Quantum calculus extends classical calculus but does not replace it. More work is needed to translate physical theories and applications to this language. While classical calculus might appear simpler in many applications, one can imagine that historically, mathematics could have developed differently in which classical calculus plays the role, the theory of distributions does here. It is always enlightening to ask "how aliens would do calculus". There are other ways to extend calculus. I consider it a matter of historical development that nonstandard analysis is not taught now in schools. Euler worked with infinitesimal and infinite quantities but it needed a long way to put this language on a precise footing.

0.2 About these notes

I see a large part of my PhD theses at ETH as dealing with quantum calculus but with a twist: in ergodic theory, a measure preserving transformation T on a probability space defines a derivative $f'(x) = (f(T(x)) - f(x))/h$. Interesting cohomological questions appear. While this cohomology is actually group cohomology, it can be interpreted as a discrete version of de Rham cohomology. In the fall of 2010, I started to change perspective and see a better generalization of de Rham cohomology which first of all is coordinate independent. Choosing a dynamical system on a set is equivalent to chose a basis and many of the notions I had contemplated with before are very much dependent on the dynamical system. Still, it is exciting because many mathematical topics can be treated with the new setup. Fluid dynamics, cellular automata, differential topology, mechanics, packing problems, integrable systems. One motivation to deal with such questions was already then to make my teaching more exciting. Even teaching very basic calculus is exciting, because one can think simultaneously on how to deal with the questions in a more general setup. And also contemplate the important question: why is it done that way? Are there no alternatives?

Kac's book renewed my motivation to look at "calculus without limits" again. In multivariable calculus, I often would often during my teaching at the end of the semester add a lecture "how would aliens do calculus?"

Despite the uniformity with which calculus is presented nowadays, there are a dozen of alternative settings to do calculus. Some are quantum calculus flavors.

I started these notes in the last week of 2006 while at Cape Cod. I proofread them a bit in August 2007 while traveling Switzerland and Israel I worked on it in the last week of 2007 again at Cape Cod.

Added 2013 while posting: I have left these notes in 2007 and followed other paths since which look very interesting, both in Riemannian geometry as well as in graph theory. Since after some Pecha-Kucha talk with quantum calculus on March 6, 2013, there was some interest in what I have written on quantum calculus, I post these scribbles. They should be understood as a notebook and are not worked out well. There are many, many lose ends and countless many typos or statements. Be warned.

Chapter 1

One dimensional calculus

1.1 The derivative

We fix a positive real constant $h > 0$. Even so there are no restrictions on the size of h in any of the following, one should think about h as a small quantity, similarly as the **Planck constant** in quantum mechanics. The reason to look at small h is that the limit $h \to 0$ recovers classical calculus. But there is no other reason. Everything can be done with $h = 1$ for example.

Definition. The **quantum derivative** is

$$f'(x) = \frac{f(x+h) - f(x)}{h} .$$

We will often use **derivative** alone and call the usual derivative the **classical derivative**.

Examples: The quantum derivative of x^3 is $((x+h)^3 - x^3)/h = 3x^2 + h3x + h^2$. The quantum derivative of e^x is $(e^{x+h} - e^x)/h = e^x(e^h - 1)$.

Here are some basic formulas and facts. They all hold for **continuous** functions. They do not need to be classically differentiable.

Theorem 1.1.1 (Differentiation formulas)**.** For any continuous function f, we have
a) the quantum Product formula: $(fg)'(x) = f'(x)g(x+h) - f(x)g'(x)$.
b) the quantum Quotient formula: $(f/g)'(x) = (fg' - f'g)/g(x)g(x+h)$.
c) the quantum Chain rule: $f(g(x))'_h = f'_k(g(x))g'_h(x)$ with $k = g(x+h) - g(x) = g'h$.

We will later see how a deformation of the algebra allows to regain familiar formulas.

Problems.
1. Is it true that if $f'(x) = 0$ on $[a, b]$ then f is constant on $[a, b]$?
Solution. No, $f'(x)$ is identically zero for any continuous h-periodic function like for example for $f(x) = \sin((2\pi/h)x)$.
2. Is it true that if f is classically differentiable and $f'(x) = 0$, then the classical derivative is zero somewhere in $[x, x+h]$?
Solution. Yes, this is the classical Rolle theorem: $f(x) = f(x+h)$ implies that there is a $x_0 \in [x, x+h]$ for which $f'(x_0) = 0$.

Remark. Quantum derivative can be defined in many different ways: One could for example take two h_1, h_2 and define

$$f'(x) = \frac{f(x+h_1) - f(x)}{2h_1} + \frac{f(x+h_2) - f(x)}{2h_2} .$$

More generally, one can define for any measure μ define

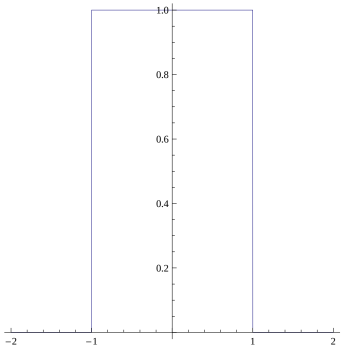
$$f'(x) = \int (f(x+h) - f(x))/h \; d\mu(h)$$

A special case is $\mu = dx$, for which f' is the Hilbert transform

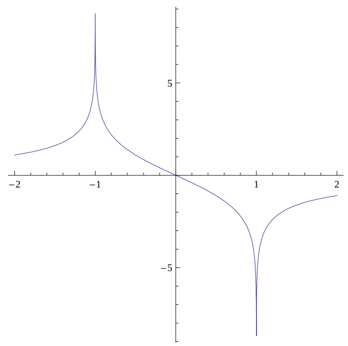
$$\int \frac{f(x+h) - f(x)}{h} \; dh$$

of f and where quantum calculus has been called "quantized calculus" by Connes.

Exercise. Does the **quantized calculus derivative** exist for $f(x) = 1_{[-1,1]}$?
Solution. The quantum derivative $(f(x+h) - f(x))/h$ is $1/h$ on $[-1-h, -1]$ and $-1/h$ on $[1-h, 1]$.



The function $1_{-1,1}$.



The quantized calculus derivative.

1.2 Integration

We have two different integrations, a discrete summation and integration over regions, which leads to integral theorems.

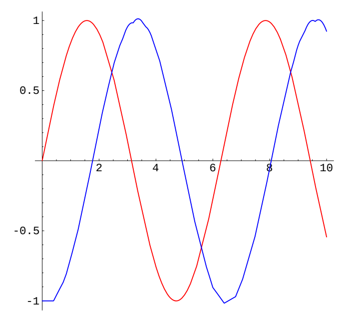
Definition. The **quantum anti-derivative** of a function f with support on an interval is defined as

$$F(x) = h \sum_{k=1}^{\infty} f(x-kh) = h[f(x-h) + f(x-2h) + \dots]$$

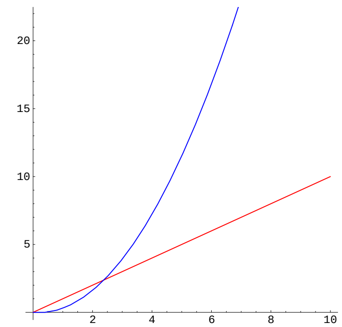
Remark. This is the analog to the classical integral $F(x) = \int_{-\infty}^{x} f(x) \; dx$.

Theorem 1.2.1 (Fundamental theorem of calculus II)**.** The anti derivative satisfies $F'(x) = f(x)$.

Proof. $F'(x) = [F(x+h) - F(x)]/h = [f(x) + f(x-h) + f(x-2h) + \dots] - [f(x-h) + f(x-2h) + \dots] = f(x)$.



The quantum anti-derivative of the classical $f(x) = \sin(x)$.



The quantum anti-derivative of $f(x) = x$.

Remarks:
1) The quantum anti-derivative is the analogue of the classical anti-derivative $\int_{-\infty}^{x} f(y) \; dy$. For a general continuous function, one can define

$$F(x) = h \sum_{k, x-kh>0} f(x-kh)$$

which is the analogue of $\int_0^x f(x) \; dx$ for the classical antiderivative.

1) The anti-derivative is determined only up to a periodic function g of period h because $(f+g)' = f'$.

Definition: the **definite integral** of f on the interval a, b is

$$\int_a^b f(x) \; dx = h \sum_{x-kh \in [a,b]} f(x-kh) .$$

Theorem 1.2.2. For $b - a = nh$, we have $\int_a^b f'(x) \; dx = f(b) - f(a)$.

Exercise: Give an example for which $b - a$ is not a multiple of h, and for which $\int_a^b f'(x) \; dx = f(b) - f(a)$ is false.
Solution. Take $f(x) = x$ and $a = 0, b = h/2$. Then $\int_a^b f'(x) \; dx = f(a+h) - f(a)$.

Theorem 1.2.3 (Partial integration)**.** Given functions f, g, let $F = \int f$ be the antiderivative. Then

$$\sum_{k=a}^{b-1} f(k)g(k) = F(k)g(k)|_a^{b-1} - \sum_{k=a}^{b-1} F(k)(g(k+1) - g(k)) .$$

Remark. We can rewrite this as $\int_a^b fg = Fg|_a^b - \int_a^b Fg'$ as in the classical case.

Proof. This is a reformulation of the Abel partial summation formula. Assume all sums from $k = a$ to $k = b-1$:

$$\sum f_k g_k + \sum F_k(g_{k+1} - g_k) = \sum (F_k - F_{k-1})g_k + \sum F_k(g_{k+1} - g_k) = \sum F_k g_{k+1} - F_{k-1} g_k = F_b g_b - F_a b_a \ .$$

Exercise:
a) Given $F = \int f$. If the sequence $F(n)b(n)$ and the integral $\int_0^x Fg'$ both converge, then $\int_0^\infty fg$ converges. b) Abel crititerium: if g is monotone and bounded and the limit $F(x)$ exists for $x \in \infty$, then the limit $\int_0^\infty fg$ exists.

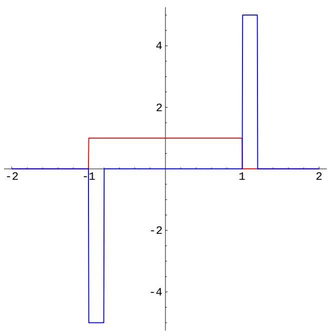
1.3 Stokes in one dimension

Definition: Quantum sets are represented by functions on the real line.

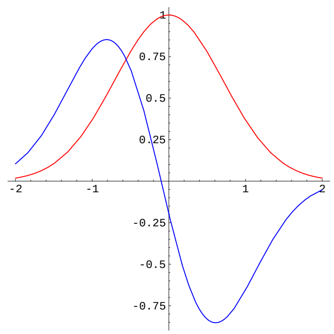
Any classical set J is a quantum set represented by its characteristic function 1_J. In quantum calculus, subsets of the real line are functions on R. They can model "fuzzy sets" and also become negative.

Definition: The **quantum boundary** of a quantum set f is the function $\delta g(x) = (g(x) - g(x-h))/h$.

The **quantum boundary** of a one-dimensional set $g = 1_I$ is defined as $\delta g(x) = (1_I(x) - 1_I(x-h))/h$.



The quantum boundary of an interval is a function with the same smoothness as the characteristic function of the interval.



The quantum derivative has the same smoothness as f. It is defined for measurable functions too.

We see that taking the boundary of a set is **dual** to the quantum derivative:

$$\begin{aligned} df(x) &= (f(x+h) - f(x))/h \\ \delta f(x) &= (f(x) - f(x-h))/h \ . \end{aligned}$$

We assume in the following that all functions satisfy

$$\int_{-\infty}^{\infty} f^2(x)\, dx < \infty \ .$$

Define

$$\int_g f = \int_{-\infty}^{\infty} f(x) g(x)\, dx \ .$$

We interpret it as the integral of the "differential form" f over the "manifold" g.

Theorem 1.3.1 (Fundamental theorem of calculus)**.** For continuous functions of compact support f, g, we have

$$\int_g f' = \int_{\delta g} f \ .$$

Proof. $\int [f(x+h) - f(x)] g(x)\, dx = \int f(x)[g(x-h) - g(x)]\, dx$. □

Advanced remark. The fundamental theorem generalizes to the case, where the real line is generalized to be a probability space X with measure P and $T(x)$ is a measure preserving invertible transformation on X. Then $df(x) = f(T(x)) - f(x))$ and $\delta f(x) = f(x) - f(T^{-1}x)$ for functions in $L^2(X, P)$.

1.4 Extrema

Definition. A **quantum critical point** of a continuous function f on the real line is a point x_0, where $f'(x_0) = 0$.

Definition. A point x_0 is a **local maximum** of a continuous function f, if there is an interval $I = [x_0 - \epsilon, x_0 + \epsilon]$ around x_0 such that $f(x_0) \geq f(x)$ for all $x \in I$. It is a **strict local maximum** if $f(x_0) > f(x)$ for all $x \in I \setminus \{x_0\}$.

Remark. To include rational functions like $1/x$, classical calculus books often include points, where f does not exist into the set of critical points. We do not do that here.

Exercise: Does $f'(x_0) = 0$ and $f''(x_0) < 0$ assure that we have a local maximum nearby?
Solution: no, there is no local maximum. We have $f(x_0) = f(x_0 + h)$. We have $f(x_0 + 2h) > f(x_0 + h)$ but it is possible that $f(x_0 - h) < f(x_0)$.

Theorem 1.4.1 ((Second derivative test))**.** If a continuous function f satisfies $f'(x_0) = 0$ and $f''(x_0 - h) < 0, f''(x) < 0$, then it has a local maximum in the interval $[x - h, x + 2h]$.

Proof. $f(x+h) = f(x)$ and $f''(x-h) = f(x+h) - 2f(x) + f(x-h) = f(x-h) - f(x) < 0$ and $f''(x) = f(x+2h) - 2f(x+h) + f(x) = f(x+2h) - f(x) < 0$. Together, we have $f(x) > f(x-h)$ and $f(x) > f(x+2h)$. The continuity of f assures that we have a local maximum on $[x - h, x + 2h]$.

Theorem 1.4.2 ((Second derivative test II))**.** If $f'(x_0 - h) < 0$ and $f'(x_0) > 0$ then
a) f has a local minimum in $[x_0 - h, x_0 + h]$, b) there exists $x_1 \in [x_0 - h, x_0 + h]$ for which $f'(x_1) = 0$.

Proof. a) The two conditions together imply that the $f(x_0) < f(x_0 - h)$ and $f(x_0) < f(x_0 + h)$ so that it is is smaller than the minimum of $f(x_0 - h)$ and $f(x_0 + h$. b) We have $f'(x_0 - h) < 0$ and $f'(x_0 + h) > 0$. The classical **intermediate value theorem** applied to the continuous function $f'(x)$ shows that there exists $x_1 \in [x_0 - h, x_0 + h]$ with $f'(x_1) = 0$.
Remark. Note that the critical point and the minimum are not the same in general.

Exercise: If the additional condition $f''(x_0) < 0$ is satisfied on $[x-h, x+2h]$, is there then a unique critical point?

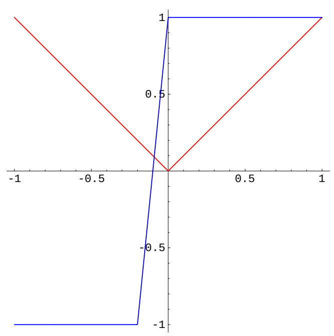
Solution: No: if $g(x) = f(x) + 1000\sin(2\pi x/h)$, then $g' = f'$ and $g'' = f''$ but g has many critical points.

Exercise: Do we need continuity? Find a bounded function $f(x)$ on $[0,1]$ such that f has no local maximum.

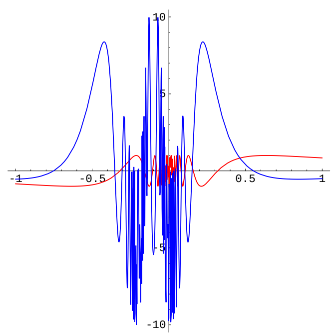
Solution. Take $f(x) = 1/(x^2+1)$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$.

Exercise: Prove that if f is smooth, then there is a classical critical point in the interval $[x_0, x_0+h]$ if the quantum derivative is zero at x_0.

Solution: This is a consequence of the classical Rolle theorem: the function satisfies $f(x_0) = f(x_0+h)$.



Example: the quantum derivative of the function $f(x) = |x|$.



Example: the quantum derivative of the function $f(x) = \sin(1/x)$.

1.5 Differential equations

In order to solve the first order differential equations $f' = g$, we solve first $f'(x) = x^n$. Then we write $f(x) = a_0 + a_1 x + \cdots + a_n x^{n+1}$ and find the coefficients from $f(x+h) - f(x) - hx^n = 0$.

$f'(x) = 1$ has the solution $f(x) = x$.
$f'(x) = x$ has the solution $f(x) = x^2/2 - hx/2$.
$f'(x) = x^2$ has the solution $f(x) = x^3/3 - 3hx^2/3 + 2h^2x/3$.
$f'(x) = x^3$ has the solution $f(x) = x^4/4 - 3hx^3/2 + 11h^2x^2/4 - 3h^3x/2$.

$f1[x_] := x$
$f2[x_] := x^2 - hx$
$f3[x_] := x^3 - 3hx^2 + 2h^2x$
$f4[x_] := x^4 - 6hx^3 + 11h^2x^2 - 6h^3x$.

Lets call $[x]^n$ the solution to $f'(x) = [x]^{n-1}$. If we can expand a function f as

$$f = \sum_n a_n [x]^n$$

then

$$f' = \sum_n a_n n [x]^{n-1} .$$

This leads to a deformed Taylor expansion of f which behaves in a similar way as the classical expansion. This follows from the product rule $d/dx[x]^n = n[x]^{n-1}$, where $[x]^0 = 1, [x]^1 = x, [x]^2 = x^2 - hx$, $[x]^3 = x^3 - 2hx^2 + ax + b$. Because $[0]^n = 0$, we have the same Taylor expansion.

Theorem 1.5.1 (Taylor expansion)**.** For any bounded function f on the real line there is a bounded function

$$f^+(x) = \sum_n a_n [x]^n$$

with $a_n = f^{(n)}(x)/n!$. We have $f(kh) = g(kh)$ for all $k \geq 0$. If f is analytic, with classical Taylor expansion in $[a,b]$, then $f(x) = f^+(x) = f(x)$ for all $x \in (a,b)$.

As a corollary, we know that $f(x) = \sum_n a_n [x]^n$ satisfies $f'(x) = \sum_n n a_n [x]^{n-1}$.
Remark. For non-analytic functions, we can add a h-periodic function to f without- changing the expansion.

1.6 Deformation of the algebra

In order to "save" $[x]^n[x]^m = [x]^{n+m}$, we can deform the algebra and define a new multiplication so that this holds. Do we have distributivity? If $f = \sum_i a_i [x]^i$ and $g = \sum_i b_j [x]^j$, then $fg = \sum_{ij} a_i b_j [x]^{i+j}$. The product rule holds now and the classical Taylor expansion too. We can also do integration by parts.

Given a classical function $f(x) = \sum_n a_n x^n$, we can define its quantum version $f(x) = \sum_n a_n [x]^n$. All the classical differentiation rules except the chain rule still hold.

1.7 The chain rule

It is possible to deform the algebra, so that the composition of functions and the chain rule $d/dx f(g(x)) = f'(g(x))g'(x)$ works? Probably not:

```
f[x_]:=x^2 - h x;   Simplify[(f[x+h]-f[x])/h==2 x]
g[t_]:=b t;         Simplify[(g[x+h]-g[x])/h==b]
A = (f[g[t+h]]-f[g[t]])/h
B = 2 g[t] b
```

Is only true for $h = 1$ or h=0. The composition of functions changes the h value, even when linear functions are involved. The quantum derivative depends on the coordinate system, because it depends on an absolute h. Do we need to carry along with different h?

The deformation of the chain rule in a discrete calculus is intrinsic in nature. It has consequences: an integrable system like the Kepler problem becomes non-integrable and produces some diffusion. KAM theory assures that for small h, most of the integrability remains.

1.8 Rolle's theorem

Is it true that if f is continuous on $[a,b]$ and $f(a) = f(b)$ as well $b - a > h$, then there exists $x \in [a,b]$ with $f'(x) = 0$?

It is false. There are functions which are $f'(x) > 0$ everywhere: with $a = 0, h = 1$ $b = [hk] + 1$. The function $f(x) = (4/5)x + \cos(2\pi x/h) - 1$ satisfies $f(0) = f(5/4) = 0$ and $f(x+h) - f(x) > 0$.
The correct quantum Rolle theorem is as follows:

Theorem 1.8.1 (Quantum Rolle theorem)**.** Assume f is a continuous function and $f = 0$ on the points $g = 1_{[a,a+h]}$ and $h = 1_{[b,b+h]}$], then there exists a point $x \in [a,b]$, where the quantum derivative f' of f is zero.

Proof. Assume the statement is false and $f'(x) \neq 0$ on $[a,b]$. Then by the continuity of f', the nonzero function f' is always positive or always negative. This contradicts the the quantum fundamental theorem of calculus which says $\int_a^b f'(x)\,dx = 0$. □

Can we improve this? How small can the points become?

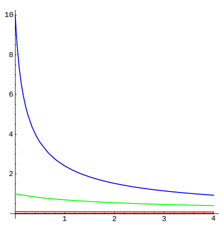
1.9 The exponential function

The function

$$f(x) = \exp_h(x) = (1+h)^{x/h} = e^{x(1-h/2+h^2/3-h^3/4\ldots)} = e^{c_h x}$$

satisfies $f(x+h) - f(x) = hf(x)$. It is therefore a solution to the quantum differential equation $f'(x) = f(x)$. We call it the **quantum exponential**. For $h \to 0$, it becomes the usual exponential function. For any C, also $C\exp_h(x)$ is a solution to the differential equation.

The quantum exponential is the classical exponential but it is squeezed by a factor $c_h = 1 - h/2 + h^2/3 - \ldots$. Define $\log_h(x)$ as the inverse of $\exp_h(x)$.



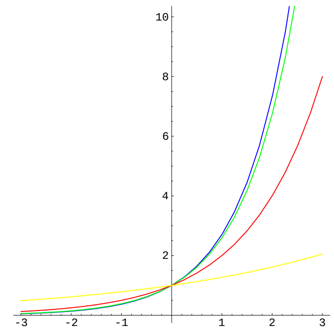
The deformation constant $c_h(a) = \log(1 + ah)/h$ for various a. We have $c_0(a) = a$, the classical case. As larger a gets, as more the quantization decreases the frequency.

There are infinitely many solutions to the above discrete difference equation. Take any function on $[0,h]$ which satisfies $f(0) = 1, f(h) = 1 + h$ and use the differential equation to define $f(x)$ on $[h, 2h]$ etc. However, there is a unique deformation if we insist on analyticity in the deformation parameter.

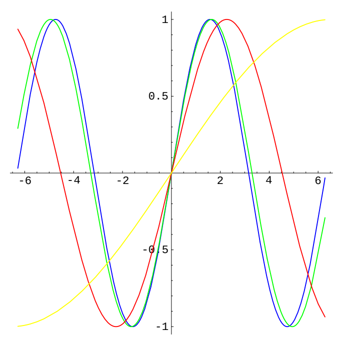
Theorem 1.9.1 (Uniqueness of exponential)**.** For any $h > 0$, the function $\exp_h(c_h(a)x)$ is the unique solution to the differential equation $f' = af, f(0) = 1$ which is analytic in h. It is also the unique solution which satisfies the functional equation $f(x+y) = f(x)f(y)$.

Proof. We have to solve the functional equation $f(x+h) = (1+h)f(x)$. Lets write $g(x) = \log(f(x))$. Then $g_h(x+h) = \log(1+h) + g_h(x)$. Write $g_h(x) = G_0(h) + xG_1(h) + \ldots$ then $g_h(x+h) = G_0(h) + (x+h)G_1(h) + \ldots$. and $g_h(x+h) - g_h(x) = hG_1(h) + (2hx + h^2)G_2(h)$, we see $G_2(h) = G_3(0) = \ldots = 0$ and $hG_1(h) = \log(1+h)$. Therefore, $g(x) = x\log(1+h)/h$.

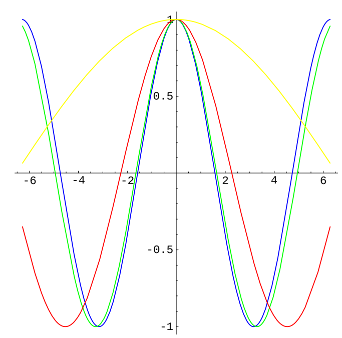
Definition. Define $\sin_h(x) = [\exp_h(ix) - \exp_h(-ix)]/(2i)$ and $\cos_h(x) = [\exp_h(ix) + \exp_h(-ix)]/2$ for which we have $\exp_h(ix) = \cos_h(x) + i\sin_h(x)$, a deformation of the usual Euler formula.



The quantum exponential for different values of $h = 10, 1, 0.1, 0.01$.



Example: the quantum sin function for $h = 10, 1, 0.1, 0.01$.



Example: the quantum cos function for $h = 10, 1, 0, 1, 0.01$.

We see that $\sin'(x) = \cos(x)$, $\cos'(x) = -\sin(x)$ as in the classical case.

1.10 Intermediate value theorem

The classical intermediate value theorem does not involve differentiation and applies in the quantum case too. Nevertheless, there is a generalization which can be considered a quantum version and merges in the limit $h \to 0$ to the classical version.

Theorem 1.10.1. If f is a bounded measurable function and $A = [a, a+h], B = [b, b+h]$ and $r = \int_A f \leq t \leq \int_B f = s$. Then, there exists $x \in [a, b]$ such that for $X = [x, x+h]$ we have $\int_X f = t$.

Proof. The function $g(x) = \int_x^{x+h} f(x)\, dx$ is continuous. Apply the classical intermediate value theorem. □

1.11 Hopital rule

Hopitals rule in the quantum case is a triviality but looks like the classical Hopital rule. If $f(x) = 0, g(x) = 0$, then

$$f'(x)/g'(x) = f(x+h)/g(x+h) .$$

You see that in the limit $h \to 0$, we get the classical Hopital rule.

Chapter 2

Multivariable calculus

2.1 Partial derivatives

Definition: Define partial derivatives as usual $f_x(x, y, z) = [f(x+h, y, z) - f(x, y, z)]/h$. The gradient of a function f is defined as $\nabla f = \langle f_x, f_y, f_z \rangle$.

Example: In three dimensions, vector fields come as one forms or two forms. They are defined as $F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$. The gradient of a function, the curl of a 1-form or the divergence of a 2-form are defined as in classical calculus: $\nabla f(x, y, z) = \frac{1}{h}\langle f(x+h, y, z) - f(x, y, z), f(x, y+h, z) - f(x, y, z), f(x, y, z+h) - f(x, y, z) \rangle$.
Definition: the directional derivative of a function f into the direction v at a point (x, y, z) is defined as

$$D_v(f) = \nabla f \cdot v$$

Clairots theorem: $f_{xy}(x, y) = f_{yx}(x, y)$.

Proof. $(f(x+h, y+h) - f(x, y+h)) - (f(x+h, y) - f(x, y)) = f(x+h, y+h) - f(x+h, y)) - (f(x, y+h) - f(x, y))$. □

Example. Let $f(x, y) = \log(3^x 2^y)$, then we have linearity in each variables $f(x+t, y) = f(x, y) + f(t, y)$ and $f(x, y+s) = f(x, y) + f(x, s)$. In that case, Clairots theorem is true.

2.2 The multivariable chain rule

The chain rule is needed at various places in multivariable calculus. For example, to justify the fundamental theorem of line integrals, the parameterization independence of arc length, that the gradient is perpendicular to level surfaces.

How much can be carried over to the quantum case? For example, is a quantum deformation of a Hamiltonian system in the plane still integrable? We have $x'(t) = H_y, y'(t) = -H_x$. The classical verification of the conservation of energy needs the chain rule. The chain rule does not survive the quantum deformation however:

$$f(g(x+h)) - f(g(x)) \neq [f(g(x)+h) - f(g(x))][g(x+h) - g(x)] .$$

The missing chain rule is not a big problem because the fundamental theorem of line integrals and arc length work without it. Surfaces are defined so that the gradient is automatically perpendicular to them. Lines are defined so that the gradient is perpendicular. This is given by the definition of the boundary as well as the definition of the exterior derivative.

Remark: A function $H(x,y)$ defines a **differentiable structure** $(h(x,y), k(x,y))$ defining the derivatives at a point so that the free evolution preserves a level curve of f and the chain rule works: define the gradients

$$\nabla_{h,k} = (f_{x,h}, f_{y,k}) .$$

Proposition: $\nabla_{x'h,y'k} f$ is perpendicular to a curve $(x(t), y(t))$ on the level curve.

The gradient depends now on the chosen curve. Should we leave the classical gradient with fixed h and let it define a Riemannian metric g such that the gradient is still perpendicular to the curve? The problem is that the metric depends on the chosen curve. We could however define a curve $(x(t), y(t))$ on the curve so that the speed is 1.

2.3 Manifolds and Forms

Before we discuss Stokes theorem in detail, we look at it in general. The concept of points, curves, surfaces and solids and the concept of differential forms are very similar in quantum calculus. They are dual to each other.

A k-**form** in n dimensions is an object

$$\sum_I f_I x_I ,$$

where the sum goes over all subsets I of $\{1, 2, ..., n\}$. and $x_I = x_1 \wedge \cdots \wedge x_k$.

A k-**manifold** in n dimensions is an object

$$\sum_I f_I x_I$$

where $x_I = x_1 \wedge \cdots \wedge x_k$.

The **hodge dual** of a form or manifold is

$$\sum_I f_{I^*} dx_{I^*}$$

where $I^* = \{1, \ldots n\} \setminus I$.

The **exterior derivative** of a k-form is the $k+1$ form $\sum_{I,i} f_I dx_i \wedge dx_I$. The **boundary** of a k-manifold is the $(k-1)$-manifold $\delta = *d*$.

Unlike in the classical case, differential forms and manifolds are mathematically the same objects.
Remark. In the case of smooth differential forms in the limit $h \to 0$, we get the classical differential forms. k-manifolds are obtained in the limit $h \to 0$ as the boundary of $(k+1)$-manifolds.
For two k-objects, define $\int_g f = \int f(x) \wedge *g(x)\, dx$. We have:

Theorem 2.3.1 (Stokes theorem)**.** $\int_g df = \int_{\delta g} f$

Proof. $\langle g, df \rangle = \langle \delta g, f \rangle$ reflects the fact that the boundary operator δ^* is the adjoint of the exterior derivative δ. □

We see that the formalism is more symmetric and transparent in the quantum case. Lets look how the general Stokes theorem looks like in small dimensions.

2.4 Curves in two dimensions

In two dimensions, curves are given as $C = Pdx + Qdy$.

The boundary of a region $fdx \wedge dy$ is the curve $\delta f = *d * f = *(f_x dx \wedge f_y dy) = -f_y dx + f_x dy$.

If f is the characteristic function of a classical region, the boundary can be visualized as a vector field near the boundary of the region oriented counter clockwise. While for $h \to 0$, the curve becomes a vector field located on the curve. For $h > 0$, the curve is fuzzy.

Theorem 2.4.1 (a)**.** For a region $fdx \wedge dy$, the boundary vector field is divergence free. The flow lines of this vector field are closed.

In other words, if we interpret the curve δG as a vector field, it is a Hamiltonian field.

If the curve vector field is compressible, then it is not the boundary of a region. By the Poincare Bendixson theorem, one can look at the long time behavior of a curve and define natural regions inside limit cycles.

A curve is given as $C = (P, Q)$ which is the boundary of a region G given by a function f. In the classical limit, a curve defines a vector field and the classical curves are integral curves of this vector field. In our case, the dynamical system $x \to x + hC(x, y)$ is a discrete dynamical system and the curve is an orbit of this system. Greens theorem relates the line integral of a vector field F along C with the integral of the curl of F over the region G. This is a continuous integral. If we look at the orbits and look at the ergodic averages, we obtain an orbit line integral. It is obtained from an invariant measure of the dynamical system. Is there a measure on G naturally associated with the measure obtained from the curve so that Greens theorem generalizes? If yes, how do we find it? Greens theorem with respect to the initial measure is

$$(\delta G, F) = (G, dF) .$$

Now we have this just weighted with a measure μ.

If we have a periodic orbit, then the line integral is a finite sum as is the integral of the curl. If we introduce a second dynamical system whose orbit fills out G and has as a boundary the finite orbit, then we have a more natural version of Greens theorem.

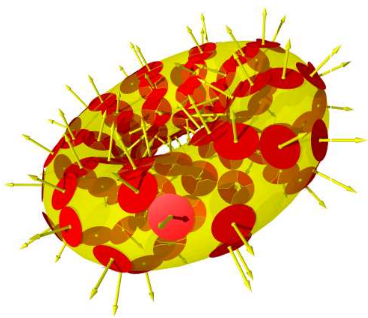
2.5 Surfaces in three dimensions

Surfaces in three dimension are described as $S = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$. One can think of it as a weighted plane field. At every point there is a plane normal to (P, Q, R) with weight the length of (P, Q, R).

If $fdx \wedge dy \wedge dz$ is a quantum region, then its boundary S consists of a plane field $S = f_x dy \wedge dz + f_y dz \wedge dx + f_z dx \wedge dy$.

Theorem 2.5.1. For a solid $fdx \wedge dy \wedge dz$, the boundary plane field S is irrotational.

Question: Are the integral surfaces closed surfaces?



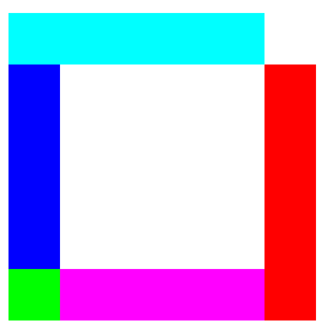
A surface in two dimensions is given by a planar field. The boundary of a solid is a planar field which is tangent to level surfaces of the function defining the solid.

Surfaces in three dimensions can be seen as orbits of a dynamical system with two dimensional time.

2.6 Line integrals

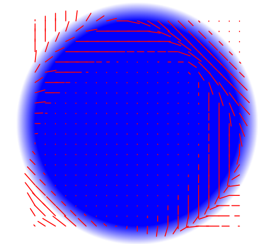
Given a 1-form F and a curve C, $\int_C F$ is the line integral of F along the curve. In the quantum version, it is an integral which does not need a parameterization of the curve because the velocity field is already given. The quantum line integral makes sense for non-smooth regions like the Koch snowflake. The limiting the line integral in such cases can be derived from the classical Green theorem too.

A square region in the plane $f = 1_{[0,1]\times[0,1]|}$



The quantum boundary of a square region in the plane.

A square region in the plane



The quantum boundary of a square region in the plane

The actual boundary curve is an integral curve of the vector field.
Given a vector field F which is irrotational in the plane. How do we find the quantum potential?

We have two different integrals, the integral in the plane and the anti derivative. The line integral is a double integral and the anti derivative is a finite sum. Greens theorem shows that the line integral of F along the boundary of a region G is zero if $F = df$.
If $F = (P, Q)$ satisfies $dF = 0$, then $F = df$ for some function. The function f can be computed using antiderivatives which are finite sums. The discrete integral along a closed loop is also zero.

Let $S(x, y, z) = \langle o, p, q\rangle$ be a surface. It can be the boundary of a region $\delta g = \langle g(x-h, y, z) - g(x, y, z), g(x, y-h, z) - g(x, y, z), g(x, y, z-h) - g(z)\rangle$ or have a boundary curve $\nabla^* \times g$. The boundary of a curve r is the function $\nabla^* \cdot r$.

2.7 Extremal problems

Definition: A point, where the gradient of f is zero is called a **critical point** of f. If a function has a quantum critical point, does this imply that there is a classical critical point nearby? We know that there are curves passing by where $f_x = 0$ and where $f_y = 0$. What happens near a classical critical point? How does the quantum gradient vector field look like and its length $g = ||\nabla f||^2$.

Exercise: Is it true that if g has a critical point, then f has a critical point.
How does the second derivative test look like for functions $f(x, y)$ of 2 variables? Lets assume a point (x, y) is a critical point Even if we assume the discrete discriminant D is positive and $f_{xx} < 0$ in a disc of radius $2h$, this does assure that we have a critical point in that disc?

Theorem 2.7.1 (Extremization of several variables)**.** If $f(x, y)$ is a continuous function of two variables on $I = [x_0 - h, x_0 + 2h] \times [y_0 - h, y_0 + 2h]$. and $|\nabla f(x, y)| < hf_{xx}(x, y)$ and $|\nabla f(x, y)| < hf_{yy}(x, y)$ in I, then f has a local minimum in I.

Lemma 2.7.2. Let f be a continuous function of one variable. If $|f'(x)| < hf''(x-h)$ and $|f'(x)| < hf''(x)$, then there exists a local minimum in the interval $[x, x+2h]$.

Proof. 1. case: $f(x+h) > f(x)$. The condition $f'(x) = |f'(x)| < hf''(x-h)$ implies $f(x-h) > f(x)$. There is a local minimum in $[x-h, x+h]$.
2. case: $f(x+h) < f(x)$. The condition $-f'(x) = |f'(x)| < hf''(x)$ implies $f(x+h) < f(x)$. There is a local minimum in $[x, x+2h]$.
3. case: $f(x+h) = f(x)$. The two conditions imply $f(x-h) > f(x) = f(x+h) < f(x+2h)$.

Proof of the theorem:

Proof. The condition implies $f_x(x,y) < hf_{xx}(x,y)$ and $f_y(x,y) < hf_{yy}(x,y)$. By the lemma, for every fixed y, there exists a $x(y)$ such that $f(x(y), y)$ is smaller than the minimum on the vertical classical boundaries. Similiarly, there exists $y(x)$ such that $f(x, y(x))$ is smaller than the minimum on the horizontal classical boundaries. Assume (x,y) is the minimum of f on the classical closed set I and that it is on the boundary. If it is on a vertical boundary, we have a point in the interior where it is smaller. Also if it is on a horizontal boundary.

Challenge: does there exist $(x_1, y_1) \in I$ for which $\nabla f(x_1, y_1) = (0,0)$? Here is an argument: the function $g_1(x,y) = f(x+h, y) - f(x,y)$ is negative on the left boundary and positive on the right boundary. There is for every y a point (x,y), for which $g_1(x,y) = 0$. This defines a curve from the left boundary to the right boundary. Similarly, there is a curve from the lower to the upper boundary where $g_2(x,y) = f(x, y+h) - f(x,y)$ is zero. The point is the intersection point of these two curves. The problem is that the curves might not be connected sets and not intersect even for smooth functions.

Remark. The minimum found in the theorem does not need to be a strict minimum. It might be part of a set on which the minimum is attained.

2.8 Lagrange

What is the analog of extrema under constraints? Is there a Lagrange multiplier method? Is there an anlogue to the Lagrange equations? What is a constraint?

Assume we want to extremize the function f on the constraint $g = c$. We can setup the discrete equations $\nabla f = \lambda \nabla g, g = c$ and solve them. This is defined for continuous functions f, g and leads to the classical limit.

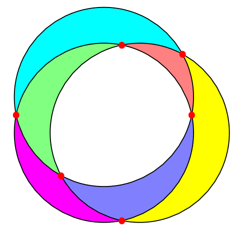
Thoughts: a) Because the boundary of $g \leq c$ s a function δg, we could also try to find the extremum of f on the region $\delta g \neq 0$. If the boundary of g has the entire plane as support, then this is the usual extremization problem. b) We might not want to find a point but a function as an extremum because points are described by functions too.

2.9 Arc length and surface area

Arc length and surface area are integrals defined as usual.
Definition. Given a curve $C = Pdx + Qdy$, define the **arc length** as

$$|C| = \int \sqrt{P^2 + Q^2}\; dxdy .$$

Example. What is the arc length of the boundary of a disc of radius r.



The quantum boundary of a disc of radius 1 for $h = 0.4$.

Definition. Given a surface $S = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$, we define its **surface area**

$$|S| = \int \sqrt{P^2 + Q^2 + R^2}\; dxdydz$$

With more general fuzzy curves, what plays the role of geodesics?
Cohomology: if the underlying space is R^3, then every vector field with vanishing curl is a gradient field. On the three dimensional torus, this is no more true. Also rationality conditions matter for h. We get into ergodic cohomology.

2.10 Partial differential equations

The **transport** or **advection** equation $f_t = f_x$ has the solution $f(t,x) = g(x+t)$, where $g(x) = f(0,x)$. This is true for all h and therefore also in the classical limit.

Since $D_x^2 - D_t^2 = (D_x - D_t)(D_x + D_t)$. We also have the general solution of the **wave equation** $f_{tt} = f_{xx}$.

2.11 Taylor formula

$D_t f = D_x f$ shows that $f(t,x) = e^{D_x t c_h} g$. This leads to a Taylor formula

$$g(x+t) = \sum_{k=0}^{\infty} (D_x^n g) g \frac{t^n c_h^n}{n!} .$$

We already have $g(x+h) = g(x) + hD_x g(x)$. Write $U = 1 + hD$. It is a unitary operator: $Uf(x) = f(x+h)$. We can write

$$f(x+t) = \exp_h(D_x t) = \exp(c_h(D_x)t) = \exp(\log(1 + D_x h)/ht) = \exp(\log(U)t/h) = U^{t/h}$$

This works formally very well.

The quantum Taylor expansion of a function f is the series $\exp(\log(1 + D_x h)/ht)f$. We have

$$\begin{aligned} \exp(\log(1 + D_x h)/ht) &= (1 + D_x t + \frac{D_x^2 t^2}{2} + \frac{D_x^3 t^3}{6} + \ldots \\ &- [\frac{D_x^2 t}{2} + \frac{D_x^3 t^2}{2} + \frac{D^4 t^3}{4} + \ldots]h \\ &+ [\frac{D_x^3 t}{3} + \frac{11 D^4 t^2}{24} + \frac{7 D_x^5 t^3}{24} + \ldots]h^2 \end{aligned}$$

It is the quantum deformation of the usual Taylor formula. For $t = h$, we get $f(x+h)$. But now, D_x is a bounded operator of norm ≤ 2. What does this give if $T(x) = x + h$ is replaced by a transformation which can not be interpolated by a flow?
For $h > 0$, the Taylor formula is formally for any bounded continuous function. But it only exists if the translation $U = 1 + Dxh$ has an infinitesimal generator L. In that case, it is just the usual Taylor formula.

Exercise: If $f(x)$ is a polynomial with $f'(x) = 0$ for all x, then f is constant. There are infinitely many f for which $f'(x) = 0$.

Chapter 3

Complex Analysis

3.1 Analytic functions

Define as in the classical case $d/dz = \frac{1}{2}(\partial_x - i\partial_y)$ and $d/\overline{dz} = \frac{1}{2}(\partial_x + i\partial_y)$. We deform the algebra to force $d/dz[z]^n = n[z]^{n-1}$. Taylors formula now still holds for polynomials.

We call a function f **analytic** if $d/\overline{dz}f = 0$. If $f(z)$ is a classical analytic function, then also the deformed function is analytic. But there are much more quantum analytic functions: take a classical analytic function $f(z)$, deform it and add an arbitrary function g which satisfies $g(x+h,y) = g(x,y+h) = g(x,y)$. Then $f+g$ is quantum analytic too.

We check that with $f = u + iv$, we have

$$\frac{\partial f}{\partial \overline{z}} = 0 \;\;\Leftrightarrow\;\; \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \; .$$

The right hand side are called the **Cauchy-Riemann differential equations**.

Quantum analytic functions do not need to be classically analytic. We call a function $u(x,y)$ **harmonic** in a region D if $\Delta f(x,y) = 0$. If u is a harmonic function, then there is a harmonic conjugate v so that $u + iv$ is analytic. Indeed, by the Cauchy-Riemann differential equations, we know the derivatives of v and since v has curl 0, the derivatives form a gradient field for which we have a potential v.

3.2 Theorem of Cauchy

Theorem of Cauchy: If f is a quantum analytic function and C is a curve which is the boundary of a region D, then

$$\int_C f(z)\, dz = 0 \; .$$

The proof is the same because the classical proof uses Greens theorem for the real and imaginary parts.
Cauchy integral formula: if D is a simply connected region and $C = \delta D$ is its boundary, then for any $w \in D$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(w)\, dw}{w - a} \; .$$

By differentiating the integral formula n times, one obtaines generalized integral formulas:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)\, dw}{(w-z)^{n+1}} \; .$$

As in the classical case, this immediatel implies the Taylor formula

$$f(z) = \sum_n f^(n)(z) z^n/n! \ .$$

(There were two different ways already to derive a Taylor formula in the quantum case: (1) directly using the differences, (2) by deforming the algebra and now (3) using discrete complex analysis.
Residue formula: define meromorphic functions, poles and then the residue. Possible picture: for a meromorphic function f and a curve C which is the boundary of a region G, we have

$$\int_C f(z)\, dz = 2\pi \int_G Res(f,z)\, dz \wedge \overline{dz}$$

Chapter 4

Differential equations

Unlike ordinary differential equations $\dot{x} = f(x)$ Quantum ordinary differential equations $d/dtx = f(x)$ always have solutions: the differential equation is a discrete dynamical systems and describes the time step.

4.1 The harmonic oscillator

Theorem 4.1.1 (Harmonic oscillator)**.** Theorem. For any $h > 0$, the general analytic solution to the quantum calculus harmonic oscillator

$$y'' = -a^2 y$$

is $y(t) = A\cos_h(c_h(a)x) + B\sin_h(c_h(a)x)$.

Proof. It all follows from the unique solution to the quantum calculus differential equation $f' = f$.
A quantum deformation of a harmonic oscillator makes the oscillator rotate slightly slower than the classical oscillator.
As in the classical case, we can now solve any system of linear quantum differential equations. The corresponding solutions are the same and can be obtained by deforming $\exp, \cos \sin$ for $h > 0$.

Note that $\exp(x)^n = \exp(nx)$. This shows that the quantum trigonometric functions satisfy the same identities. For example, the double angle formulas

$$\cos_h(2x) = \cos_h(x)^2 - \sin_h(x)^2, \sin_h(2x) = 2\sin_h(x)\cos_h(x)$$

both hold.

4.2 PDE's

Also many partial differential equations have solutions. Whenever it is possible to seperate one variable and write $f_t = D(f)$, where D is a difference operator in the other variables, it is possible to de

Chapter 5

Explorations

5.1 Introduction

Classical calculus is very successful in describing nature. Why would one want to explore alternatives then? As usual, simply because we can. But it remains a legitimite question, whether classical theories which are successful for example in physics, can be replaced by a discrete version without limits.
Ernst Specker asked once how it comes that we model phenomena with the language of differentials and then discretize this again to compute things. For example, waves in the ocean which are a huge discrete particle system are modeled by smooth differential equations which then when computing things are discretized again. Is there a shortcut or is the classical limit a real simplification?

5.2 Hamiltonian systems

Is there a natural way to discretize a Hamiltonian system to leave the energy surface intact? We would like to discretize systems like Kepler's problem without losing integrability. If such a system can not be modeled easily, the approach is questionable.
In the multivariable case, we would have to adapt

$$\begin{aligned} d/dt f(x(t),y(t)) &= [f(x(t+h),y(t+h)) - f(x(t),y(t))]/h \\ &= [f(x(t+h),y(t+h)) - f(x(t),y(t+h))]/h \\ +[f(x(t),y(t+h)) - f(x(t),y(t))]/h & \\ &= f_{x,x'h}(x(t),y(t+h))x'(t) + f_{y,y'h}(x(t),y(t))y'(t) \; . \end{aligned}$$

Because for a classical Hamiltonian system

$$\begin{aligned} \dot{x} &= H_y \\ \dot{y} &= -H_x^+ \end{aligned}$$

the speed is the length of the gradient vector of H, we could define in two dimensions the map $T(x,y)$ by drawing a circle of radius $||\nabla H||$ around a point and intersecting it with the level curve. This preservs the level curves and volume, if we do the steps one after each other.
For a central body problem we have with a deformed cross product a conservation of the momentum. In our case, we might have to deform the cross product on each point differenty. As long as x'' is parallel to x, we are fine.

5.3 Electromagnetism

The classical Maxwell equations $dF = 0, \delta F = j$ in 4 dimensions look the same in quantum version. If we acept that every 2-form can also be interpreted as a "two dimensional surface" and a 1-form as a "1-dimensional curve", then the Maxwell equations tell that the electromagnetic field F is a surface which has as a boundary the electric current j.

5.4 Quantum quantum mechanics

It is possible to describe classical quantum mechanics with quantum calculus. A symmetric discretization $\dot{f}(t) = f(t+h) - f(t-h) = (U - U^*)f$ of the time evolution has the advantage that with $V = iU$ we have $i\dot{f}(t) = (V + V^*)f$ so that the Schrödinger equation $i\dot{\psi} = L\psi$ becomes a discretized wave equation

$$(V + V^*)\psi = \sum_i a_i(U_i + U_i^*)\psi \; .$$

The discretisation has rendered the problem invariant under discrete Lorentz transformations. This notation includes both the case of a free particle where all U_i are dynamical unitary operators $U_i(f) = f(T_i)$ and the case of a Schrödinger operator where part of the right hand side are multiplication operators $U_i(f) = e^{inx}f(x)$ and the sum represents a cos Fourier series. With $U_0 = -V$, we even can write the Schrödinger equation in the discrete as

$$H\psi = \sum_{i=0}^{n} a_i(U_i + U_i^*)\psi = 0 \; .$$

In an ergodic setup, where the time evolution is also given by a measure preserving transformation, we just need to find an eigenvalue of the space time operator H to get explicit solutions of the Schrödinger evolution. Not every initial condition leads to an evolution which works. There is just one evolution in general. The U_i for $i > 0$ have to be invariant under the transformations T_0 if we want to model a time independent Schrödinger evolution and a unitary time evolution.

Nonrelativistic quantum dynamics deals with the Schrödinger dynamical system $i\hbar\dot{\psi} = L\psi$ in a Hilbert space $\mathcal{H}$. If L is a bounded difference operator, the discrete time version

$$i\frac{\hbar}{2\epsilon}[\psi(t+\epsilon) - \psi(t-\epsilon)] = L\psi(t)$$

defines a unitary evolution and is useful for studiing spectral measures of L ([?]). If the left hand side is written as $ia^{-1}(U - U^*)$ and $V = iU$, then this is $V + V^* = aL$ a discretisation of a relativistic wave equation. This discretized dynamics is the dynamics of an operator $\tilde{L}$ which satisfies $\cos(\tilde{L}) = aL$ with $||aL|| < 1$. The time evolution can then be computed by iterating $A : (\psi, \phi) \mapsto (2aL\psi - \phi, \psi)$ on $H \oplus H$. The dynamics of A has the same dynamical properties than the actual Schrödinger evolution with the modified operator. Given $\psi \in H$. Then $\psi(n)$, obtained from $A^n(\psi, 0) = (\psi(n), \psi(n-1))$ solves $\psi(n+1) + \psi(n-1) = 2L\psi(n)$.

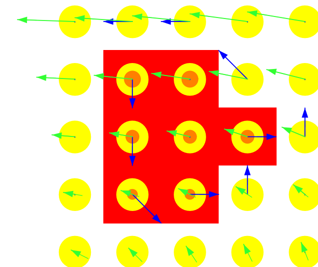
5.5 Quantization operator

Classical mechanics is described by a flow X_t and equivalently by a unitary flow $U_t f = f(X_t)$ if there is a finite invariant measure. If X_t has an infinitesimal generator D, we can define a quantum Hamiltonian $-D^2$ and the unitary evolution e^{-ihD^2}. So, the transition from classical to quantum is $U \to e^{ih(U+U^*)}$ which is on the lie algebra level $L \to i2h\cos(L)$. By diagonalization we have to understand also in the quantization operator case, the complex dynamical system $f_h(z) = ih\cos(z)$. If we are interested in iterating this "quantization map", we can look at the Mandelbrot set, the set of h, for which 0 is attractive.

The word "quantization" is used in different meanings. One is to replace a linear differential operator by a quadratic. A second is to quantize space and time. If we quantize the time evolution of the Schroedinger equation, we get an evolution $U_t = e^{ic_h(L)}$. It solves $[f(t+h) - f(t)]/h = Lf$. But how do we define the infinitesimal generator $c_h(L) = \log(1 + Lh)/h$?
The Stokes formula makes sense for any Hilbert space H and d commuting unitaries $U_1, ..., U_d$.

5.6 Generalisations

Finite set The formalism does make sense on a finite set with d commuting transformations T_i. Stokes theorem still holds. Here is an example: take the space $X = \{1,2,3,4,5\} \times \{1,2,3,4,5\}$ with 25 points.



Greens theorem on a set with 25 elements. The region has 7 points which has 10 boundary points. The vector field is $F(x,y) = (-y - y^2, x)$.

Algebra Given an algebra X with trace with d commuting automorphisms.
Ergodic case T_i are commuting measure preserving transformations. We get ergodic cohomology problems.
Changing the target space Let the functions take values in an abelian group. In the case, when it is the complex plane, then we can modify the formalism so that the boundary of a set is a complex valued function. Greens theorem is then nothing else then Cauchys theorem.
Fuzzy set theory The formalize clearly includes fuzzy calculus, where sets are given by functions.
Random version We can change the target space and still have the same theorem. Also possible are random versions, where the transformations T_i are random walks and results are obtained by taking expectations.
Noncommuting case If T_i do no more commute, then Clairot does no more hold. We have some "curvature". Can we save Stokes theorem? We need probably the adjust the notion of boundary. Can we deform the exterior derivative to get $d^2 = 0$?

5.7 Conclusion

In the discrete, calculus becomes more symmetric and elegant. Symmetries like rotational symmetries or Lorentz invariance have to be replaced by discrete versions or could be implemented additionally by finding a representation of the group on the probability space extending the discrete operators. This would actually lead to a classical Schrödinger evolution. An extension looks possible if each of the unitary operators have infinitesimal generators which commute.