ELEMENTS OF FINITE GEOMETRY

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Footnotes to Book I

Here are some remarks and references to the text. We decided to place all references at the end so that units compile separately without invoking bibtex and in order to prevent overlap. Brevity of the main text is the goal. There is a self-imposed limit of 2 pages per unit and a one minute limit for the accompanying videos. Mark Twain once said: "I didn't have time to write a short letter, so I wrote a long one instead." For me personally, brevity is a source for clarity, both for math as well as for coding. It still needs time: I spent about a dozen hours of focussed work in average for each unit, each done during one day, with 4 focussed sprints, starting at home in the morning, a coffee shop part until noon, an other coffee shop part after an outdoor run and an afternoon part in the office recording. Text was written partly during the earlier summer during travels. The footnote aims less at optimizing length and therefore meander more. Possible in the future could be an appendix with illustrations and exercises. But it would be more important to work on the next batch of 12 units.

Unit 0: Introduction

13.1. The introduction contains some motivation on why to focus on finite structures. It is a reaction to the incompleteness theorem [34]. Popularized masterfully in [43], it was one of my pivotal high school experiences that drew me to mathematics. I got exposed to "strict finitism" by Erwin Engeler at ETH (from whom I took "topology" and "mathematical software" courses). He mentioned strict finitism once in person but [25] is one of Engeler's articles from 1971. Ernst Specker (from whom I took two semesters of linear algebra, as well as logic and model theory courses and also participated in various seminars, informed the class once about the work of Kustaanheimo [99], an attempt to do mathematics in finite fields, rather than the real numbers. Kustaanheimo collaborated with Eduard Stiefel and was in Zuerich in 1964, certainly knowing Specker, as Specker and Stiefel both were students of Heinz Hopf. Peter Läuchli, a student of Specker (from whom I took two semesters of "calculus" (both single and multi-variable) and a logic course about forcing) gave later a mesmerizing course on non-standard analysis based on Nelson's internal set theory approach [107, 108]. See [65] for an other personal story about Specker and Läuchli (and other ETH teachers). The textbook that Läuchli recommended for his course was [116] and he followed it partly. IST an effective link between finitist and traditional mathematics as it just changes language and not structures. Finally, finitism is a honest reaction to the fact that all data we ever acquire and process is finite. It is a honest computer science approach to mathematics.

13.2. The Taylor formula mentioned in the introduction appeared in the Pecha-Kucha talk (a presentation in which 20 slides are shown with a strict 20 second limit to talk for each) [64]. The discrete Taylor theorem is attributed to Gregory, a contemporary of Newton. One can find discrete calculus also in the original writing of Leibniz. I used discrete calculus excessively in single variable Math 1a courses from 2011 to 2024. I was fortunate to have been given the freedom to experiment with a fresh take on single variable calculus then. In the summer of 2026, I got a used copy of van der Waerden's book [128] and found the formula on page 69 in the context of interpolation. The topics discussed in these notes (and beyond) had previously attempted to be summarized: [59, 61, 63, 73, 94, 95, 96].

Unit 1: Gauss-Bonnet

13.3. For the discrete Gauss-Bonnet theorem, see [22, 102, 54, 76, 53, 67, 72]. Lewitt did not discuss it related to Gauss-Bonnet. I learned about that paper only after [53]. The general formula was only obtained by painfully going through small dimensional manifold cases, where Dehn-Sommerville relations obscure a bit the patterns. For me the most important problem then had been to explain why I see zero curvature for all odd dimensional manifolds and this turned out to be a central and rich problem. It motivated for example to look at integral geometric approaches and the Dehn-Sommerville story. The integral geometric connection proves that the curvature is the real deal. It is not just some "analogy". It produces the Gauss-Bonnet-Chern integrand if we take a discrete approximation of a compact Riemannian manifold. Related more general curvatures appeared in [42, 92]. For the classical case, see [132]. The classical Gauss-Bonnet-Chern evolved over decades: [44, 3, 27, 2, 12, 13, 16] Discretisations of space have appeared before Poincaré, but less for general manifolds. Historically one has focussed more on polytopes [133, 39] and the monsters which Lakatos described has made geometers more careful, when talking about polytopes so that they mostly limited to convex objects which of course refers to the continuum. If we discard the continuum, we need to be extra clear in order not to repeat the scandal which Lakatos described. For history see [21]. Davadoss called the history of polytopes a delicate task. For discrete geometries see [19, 114, 105] for computational geometry [20, 4], for integrable systems [7], for computer graphics or computational-numerical methods [18, 110].

Unit 2: Poincaré-Hopf

13.4. The discrete Poincaré-Hopf appeared first in [55]. It was made use of in [83]. The work with Josellis in [51] led to simplifications. The parametrized version appeared in [79]. More elaborations came in [78] and [80]. A nice poetic approach to numbers is by seeing them geometrically [69]. I found out later that seeing numbers as a cell complex was not the first [6] but [69] still allows to interpret the Riemann hypothesis (the Mertens connection) from a topological point of view. For an early discrete version in two dimensions based on an embedding in a surfaces, see [33]. I learned about Glass'theorem from Kate Perkins thesis at Harvey Mudd [111]. The speed with which the simplex generating function can be computed recursively using the formula given in the text needs to be studied more. Computing the number of simplices in a graph is known to be a hard problem. What the recursive formula does is relegate the problem to subgraphs in unit spheres. About the classical story: Poincaré proved the index theorem in chapter VIII of [112]. Hopf extended it to arbitrary dimensions in [45]. The collection [48] contains in particular an article on the history of graph theory. A story about Euler characteristic and polyhedra is told in [115]. See [41, 125, 117] for notations in algebraic topology, [40, 5, 8] for graph theory.

Unit 3: Index expectation

13.5. The Buffon needle problem, stated and solved in 1777, is one of the first problems in the theory of geometric probability. The subject is also called integral geometry. $L = \pi \mathrm{E}[n]$ gives the length if lines are distance 2 apart. See [119]. Crofton's book on local probability is [15]. More general references on integral geometry are [118, 52, 123]. As for index expectation, see [57] [56], [75] [86] [87]. For the Nash embedding theorem source, see [98]. Gauss-Bonnet-Chern is older than Chern, if one allows embeddings. The first intrinsic proof of GBC without embedding the manifold is [12]. For historical remarks see [13]. I myself learned GBC first from Patodi's proof [16]. Integral geometry is extremely useful as it links the discrete with the continuum. Because Poincaré-Hopf indices are divisors, there is a direct link between the discrete and continuum. Integral geometry allows to bypass any discrete Riemannian manifold set-ups. Distances can be implemented using embeddings for example.

Unit 4: Euler's Gem

13.6. The task to define what a manifold is combinatorially boils down essentially to the task to characterize spheres, a question going back to Herman Weyl [130], at a time when the Grundlagen crisis had been raging still. (There is still a Grundlagenkrise today of course but similarly as the possibility of a nuclear armagedon danger is blended out from consciousness, mathematics ignores the possibility of an inconsistency to occur and even doubles down by using structures beyond ZFC like Grothendieck universes. There are two important early approaches to discrete manifolds. One approach is by Forman [28, 29, 30, 31] who built "discrete Morse theory", the other by Ivashchenko (Evako) [47, 11] whose work is sometimes placed into a field called "digital topology". Forman defines spheres Morse theoretically using functions, Evako uses induction. The compact version done here has been developed by us, first in the graph theoretical set-up, later in abstract simplicial complexes using the Alexandroff topology. What is important to make the definition decidable is to use "contractibility" (sometimes called "collapsible") and not "homotopic to 1" in the definition of a sphere. With the later, one would run into computer science trouble: there would be no Turing machine which could decide whether a given structure is a manifold or not. With "contractibility" as defined, there is a definite algorithm which terminates polynomial time once the dimension is fixed and decides whether a given graph or a given finite abstract simplicial complex is a manifold or not. Discrete Reeb [81] shows equivalence of Morse and Recursive approaches. A good general reference for the topic of Euler's gem is [115]. I gave a mathtable talk about this on 2/6/2018 to honor Euler's day 2/7/18. The innocent looking Euler Gem formula is infamous for having been proven wrong repetitively, and not by amateur mathematicians or but by the best mathematicians in the world. It so has become a show case for the epistemology of mathematical discovery [100]. I myself disagree with Lakatos that science whould be an evoluationary approach to truth. It should be true from the beginning. If this is not possible, the culprit usually is that the definitions are not good. The algebra of the join operation goes back to [134] in graph theory. It might even have appeared already in the Principia Mathematica from 1910, but that is a bit of a stretch. The analog for simplicial complexes is to take $A \cup B \cup \{a \cup b, a \in A, b \in B\}$. It produces a monoid structure on complexes and spheres are a sub-monoid. Also Dehn-Sommerville spaces form a submonoid. The classification of regular polytopes is usually done in the continuum in the context of polytopes [133, 39], which invoke Euclidean space and convexity. [38] discussed the difficulty with the term "polytop". For [14],a polytop is defined as as a convex body with polygonal faces. For source work, [122, 124, 113]. Abstract simplicial complexes appeared in 1907 [17]. See [10, 106]. [1] uses the name **unrestricted skeleton complex**. In [131] there are called **symbolic complexes**. For references on simplicial complexes, see [50, 126, 127] but definitions can vary. Especially the treatment of $\emptyset$. Insisting on having simplicial complexes not contain $\emptyset$ is much, much more natural and makes the theory look like the theory we all know when studying differential topology or differential geometry.

Unit 5: Euler-Poincare

13.7. Finite exterior calculus has been used already by the founders of algebraic topology. Discrete Hodge theory appeared first in [23]. Discrete McKean-Singer appeared in [58]. Taking simple ideas going back to Dirac and Hodge (office neighbors who did not talk much together as Atiyah once happily pointed out as he could build a career combining their ideas). Discrete calculus has been reinvented again and again, in later time mostly in applied or computer science literature. [24]. The main definition of using linear algebra in topology have already been forged by Poincarré or Betti. The matrix D is usually given in terms of its blocks, which are called **incidence matrices**. It would have been so much easier already at the time of Poincare to combine these matrices to a common matrix d on all forms and to define $D = d + d^*$ and then study the matrix D^2. The insight that one do not have to look at anti-symmetric functions, but just can look at functions, once an orientation is fixed on simplices, is probably due to Whitney who used this especially in the context of the cup product. To bypass the chain complex notations and use linear algebra is due to Hodge. It is such a relieve especially when working with computer algebra implementations not to have to deal with equivalence classses, but simply with null-spaces of matrices. The concept of the square root of the Laplacian is due to Dirac. Clifford algebra constructions show that we can bypass awkward Dirac matrix constructions. The Hodge definition is more intuitive than equivalence classes "closed forms modulo exact forms". It is a good (but maybe a bit challenging) exercice for an introduction linear algebra course to show that the two pictures equivalent. About the notation, we use H for Hodge and because we want to use L for the connection Laplacian. We write **Dirac operator** and not **Dirac matrix** in order not to confuse with Dirac matrices in mathematical physics which are used to get the square root of the Laplacian.

Unit 6: Unimodularity

13.8. The result was discovered in December 2015 while experimenting with quadratic cohomology. Neither any discrete nor continuum analog of this story seems have been found before. It was presented on a mathtable talk "Bowen-Lanford Zeta function" on 10/18/2016. The proof appeared in [68]. A reason for the delay was a summer excursion into number theory [70]. The result still surprises me today. A more direct alternative proof of unimodularity was given in [104]. I still feel the unimodularity result is remarkable and I'm glad I spent some effort to have it published [83]. Polishing something for an actual publication uses a lot of time and often is a fool's errand anyway, if one deals with a topic, that has not been covered before. One definitely has to be lucky with referees. There were various generalizations [84], [77], [82], and [85]. For some spectral problems in the context of Dirac and connection matrices, see [97].

Unit 7: Lefschetz fixed point formula

13.9. The discrete Brouwer Lefschetz theorem appeared in [60]. My proof there was close to Hopf's approach [46] which also is in [1] (Alexandrov and Hopf had collaborated). Textbooks cover it in [49, 125, 36, 35]. A version in dimension 1 is the Nowakowski-Rival fixed edge theorem [109] and a generalization [26] to a commutative family of such maps. For the classical Lefschetz fixed point theorem and Brouwer fixed point theorem see [101, 9]. The Brouwer fixed point theorem is usually stated for balls in Euclidean space, where one can prove it also with discrete methods, like Gale's proof using q-dimensional Hex games.

Unit 8: Sphere formula

13.10. The sphere formula is also a show case how to go from graphs to simplicial complexes. In both cases there are "unit spheres". For graphs, it is the graph generated by the neighbors. For simplicial complexes, the unit spheres are the boundaries of the smallest open sets. For traditional Euler characteristic, the sphere formula follows from the Green formula $\chi(G) = \sum_x \omega(x)\chi(U(x))$ which is $\mathrm{str}(L)$. One also has $\chi(B(x)) = 1$ so that $\chi(G) = \sum_x \omega(x)\chi(B(x))$ Together with the definition $\chi(G) = \sum_x \omega(x)$ this proves the sphere formula. The sphere formula immediately shows that odd dimensional manifold have Euler characteristic zero. [83, 90]. The Hydrogen relation was also discussed in [72] and [74] where it is shown that $g = \log(f)$ has the property that $g'(1)$ is the average simplex cardinality.

Unit 9: Level sets

13.11. Much of calculus or algebraic geometry or differential topology deals with level sets of functions. Arthur Sard [120, 121] and Anthony Morse [103] established that a smooth map f from a manifold M to a manifold N has the property that for almost all $y \in \mathrm{ran}(f)$, the level set $\{x \in M, f(x) = y\}$ is a manifold. Already simple examples of polynomial maps like $x^2 - y^2 = 0$ show that level sets are in general not manifolds. It can come a bit as a shock that in the discrete singularities do not occur. Level sets of any function always are manifolds if not empty. Of course, we need to define level set properly. This was done in [66] in 2015. It was generalized in 2024 [93]. Working with Jennifer Gao who wrote a thesis in that area helped [32]. The level set result get for teaching purposes in Math 136 differential geometry classes simplified. In [95], it was generalized to Dehn-Sommerville manifolds, an approach to manifolds which is recursive but only makes an assumption about Euler characteristics.

Unit 10: Index formula

13.12. The formula first appeared in [56] in 2012 and [62] in 2013. There was no level set construction yet and things were more complicated and ugly from a current perspective. It was an application for [57] but it required to define a level set structure defined by a function. It was valuable time however to work on this because it enabled the level set result three years later. I like the index formula because it is may be the swiftest way to see why odd-dimensional manifolds are flat. This observation was one of the earliest I made and it bothered me in 2010 that while possible to verify it for 3-manifolds, I had been unable to verify it for 5 manifolds already. It is also intuitive to see curvature as a genus expectation of random submanifolds in unit spheres. This shows that if we take a 4 manifold, a small sphere $S_r(x)$ in it and a random function, then the expected genus of contour surfaces $\{f(y) = f(x)\}$ is curvature. Since this is related to curvature of random 2-manifolds, this is conceptionally closely related to scalar curvature which is the expected sectional curvature at a point. It motivates to look at Euler characteristic as a serious functional with possible physical relevance. The index formula is non-trivial already for 2-manifolds G. Put random function values on the vertices of a 2-manifold and at each node v look at the average number $X(v)$ of sign changes of $f(w) - f(v)$ when w goes along the circle around v. Then $K(v) = 1 - E[X(v)]/2$ is equal to $1 - d(v)/6$, the Eberhard curvature.

Unit 11: Quadratic cohomology

13.13. This started in [67] in 2016 and cohomology in [71], where the terminology "Wu cohomology" or "connection cohomology" was used. Wen-Tsun Wu in 1953 first considered this multi-linear valuation [129]. [37] picked it up and conjectured the existence of Dehn-Sommerville invariants, which was confirmed in [67]. Tamas Reti pointed out in 2016 that for a triangle free graph with $n = f_0$ vertices and $m = f_1$ edges, we have $\chi_2(G) = n -$

$5m + M$, where M is the first **Zagreb index** $\sum_v d(v)^2$. When the fusion inequality for simplicial cohomology [89] was generalized to higher cohomology, [91], the name "quadratic cohomology" stuck. We mentioned the theory last in [97] in the context of Birkhoff's fixed point theorem. I gave a table talk on March 8, 2016 about Wu characteristic. It needs to be studied much more especially also in the context of the fusion story, where the cohomology is used also for open sets and interface cohomologies between an open and closed set are used. One can look at the quadratic cohomology for the complement of a path in a 3-sphere for example. Does this give interesting knot invariants?

Unit 12: Higher Green

13.14. This section was covered [88]. It generalizes the energy theorem in the case when $k = 2$ and $m = 1$, the Euler characteristic case. The case $k = 1$ has appeared as $\mathrm{str}(g) = \chi(G)$. It is remarkable, how prominently Green functions appear in physics. One usually associates Green functions with the inverse of Laplacians but this is only the 2-particle story. In the continuum, it leads to technical difficulties as the Laplacians are not invertible. Particle interaction potentials have singularities. The Green function of the Laplacian in Euclidean space $\mathbb{R}^3$ for example leads to the $1/r$ potential energy, we see in physics, like in electro magnetism or gravity. Every geometric space defines so a potential, in $\mathbb{R}^2$ it would be log in $\mathbb{R}^4$, two celestial bodies would interact with a $1/r^2$ potential. The work on higher Green functions was motivated by physics. In particle physics, one looks at **k-point functions** $\langle \Omega, \phi(x_1) \dots \phi(x_k), \Omega \rangle$ or Green functions to calculate the out come of particle interactions. This is usually done using a series expansion and estimated using Feynman diagrams. But in the discrete, there are no infinities, everything is finite. The 1-point function $k = 1$ case can be thought of as **vacuum expectation**. The 2-point functions $k = 2$ could have relations with **probability amplitudes** for particles to go from x_1 to x_2. The general case k could relate to **scattering amplitudes** of k particles. It is important to note however that all what is done here is just geometry and mathematics, actually combinatorics. Physics is motivation however.

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