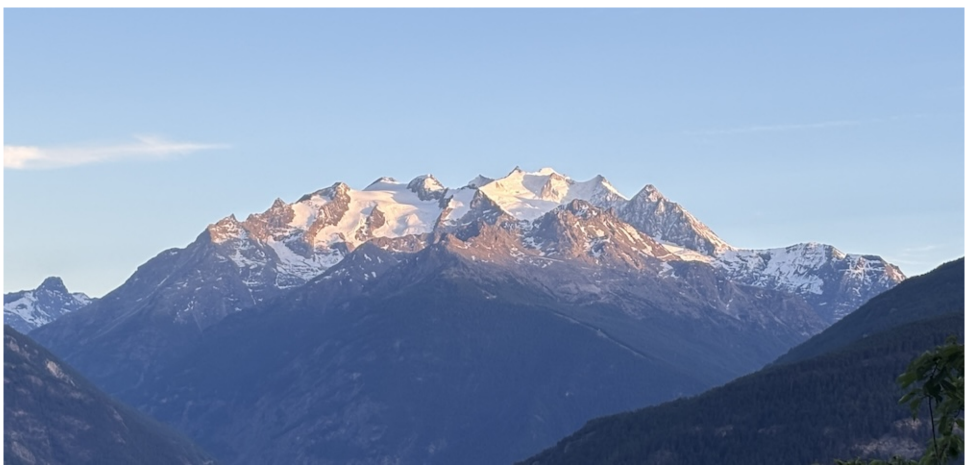
ELEMENTS OF FINITE GEOMETRY I

OLIVER KNILL



2026

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Introduction

There is a fundamental possibility that the common axiom system in mathematics is inconsistent. A strong enough foundation of mathematics like the Zermelo-Fraenkel frame work can never be proven to be consistent within itself and is always extendable in the sense that there are statements that can not be proven nor dis-proven within the system. Hilbert's dream of building a provably consistent strong theory can not be realized. The incompleteness theorems have entered popular culture especially with Hofstadter's book "Gödel-Escher-Bach". While most mathematicians - including myself - do not worry about a possible inconsistency dooms-day, the possibility of such an event is a strong motivation to pursue geometry using weaker tools. Working with finite mathematics is one of these approaches. Finite descriptions of spaces like manifolds have already been initiated in Poincaré's "analysis situs" program, but most topologists still assume objects realized in an Euclidean space. Even Dehn and Heegaard who initiated abstract simplicial complexes in 1907 used the continuum.

Finite mathematics does not necessarily escape Gödel. But it is conceivable that ZF is inconsistent while an axiom system without infinity will remain consistent. When working in a finite geometric set, we can in principle be certain to have consistency within that world. Most computer scientist are by nature more radical. They are "strict finitists" in the sense that they only work with structures for which there are tools to handle them. We can make sense of the symbol $GP = 10^{(10^{100})}$, but not of the number "googolplex" itself. As computer scientists, we am essentially blind for answering questions like how many primes there are in $[GP-100, GP+100]$. Working with finite structures puts mathematics on safer grounds. It is a refuge to which we might have to retreat one day, if some inconsistency in ZF should emerge. I personally think this can happen but that there is little to worry: finite tools will be able to emulate all continuum math. It is still possible that even after removing the infinity axiom, a catastrophic collapse could occur, but that is less likely; it would mean that even the algebra of finite sets is inconsistent. Assuming consistency of finite systems requires some faith, similarly as we trust the Turing-Church thesis or assume that our memory in the brain is reliable, allowing us to do logical steps without forgetting previous assumptions, while doing logical conclusions. In the last couple of years we all have seen intrinsically random probabilistic thinking entities to emerge that have started to compete with human thought. Aristotle would turn in his grave learning that

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humanity started to flip coins in order to draw conclusions. Probabilistic reasoning is good enough for practical purposes, but it is philosophically unacceptable, even if the probability to be wrong is only $1/GP$.

A "finite geometry" is a finite geometric structure like a simplicial complex, a graph or a delta sets. These combinatorial structures carry natural arithmetic and topological operations. One can add or multiply such structures in various ways for example. The choice of using these three structures a bit arbitrary but it is motivated by the Unix trilogy "simplicity, clarity and generality": simplicial complexes have only one axiom and so are simple, graphs are intuitive and visual and so very clear. Delta sets finally are the most general. Every graph defines a simplicial complex and every simplicial complex is naturally a delta set. Delta sets are sets of sets that admit a calculus structure in the form of face maps. Their advantage of the delta set category is that has no limitations but still allows to do all geometry that we know from in the continuum. For example: submanifolds or products of geometric structures are often a delta set at first before modeled again as a graph. Multi-graphs like quivers are delta sets. We work in this trinity.

It is tempting to label a discussion about finite geometry with terms like "quantum geometry". The meaning of the word "quantum" has been washed out recently and so has become ambiguous, even misleading. The term "quantum gravity" illustrates it. Classical geometry or calculus can be "quantized" in various ways: there are non-commutative geometry flavors, Poisson brackets can be replaced by commutators or Lagrange variational problem are deformed to path integral quantization. Every numerical scheme is some sort of quantization, as the continuum is modeled using finite structures suitable for numerical purposes.

Finite geometry is related to "calculus without limits". For example, every function f on the real line satisfies the exact Taylor formula $f(a+x) = \sum_{n=0}^{\infty} f^{(n)}(a)x^n/n!$, for integer x, if the derivative is deformed to $Df(a) = f'(a) = f(a+1) - f(a)$ and the polynomial algebra is deformed to $x^n = x(x-1)\dots(x-n-1)$. No regularity whatsoever is needed and Taylor is a finite sum. For $x = 2$ for example $f(2) = f(0)+f'(0)2/1!+f''(0)2(2-1)/2! = f(0)+2f(1)-2f(0)+(f(2)-2f(1)+f(0)) = f(2)$. If the unit 1 is the Planck scale, finite calculus produces the same physics as traditional calculus. The derivative D generates translation because $\exp(Dt)f(x) = f(x+t)$ and $Xf(x) = xf(x-1)$ and leads with the momentum operator $P = iD$ to the canonical anti-commutation relations $[X,P] = (XP - PX) = i$.

Theorems are at the heart of mathematics. A theory needs to be judged by the quality of theorems it can produce. Ideally, definitions are short and clear, theorems are simple, proofs are intelligible and examples are rich. In a first batch we cover 12 results in finite geometry. I hope to be able to continue with the next batch of 12 at a later time.

Unit 1: Gauss-Bonnet

The Gauss Bonnet theorem equates the total curvature of a geometry with its Euler characteristic. Curvature is the energy of the fundamental units of space pushed to points.

1.1. A simple model for geometry is a **finite simple graph** (V, E). Its complete subgraphs define a finite abstract simplicial complex G, a finite set of non-empty sets that is closed under the operation of taking non-empty subsets. The elements in G are known as **simplices**, **cliques** or **faces**. The empty set $\emptyset$ is not a simplex. But the empty set is called the **void** and is a simplicial complex. A simplex x of $(n+1)$ points has **dimension** $\dim(x) = n$. Its **energy** is $\omega(x) = (-1)^{\dim(x)}$. The total energy $\chi(G) = \sum_{x \in G} \omega(x)$ is the **Euler characteristic** (V, E).

1.2. Let $f_k(G)$ denote the number of elements $x \in G$ with dimension k. For example, $f_0(G) = |V|$ is the number of vertices K_1, $f_1(G) = |E|$ the number of edges K_2 and $f_2(G)$ is the number of triangles K_3 in the graph. We have $\chi(G) = \sum_{k=0}^{q} (-1)^k f_k(G)$, where q is the **maximal dimension.** The vector $(f_0(G), f_1(G), \dots, f_q(G))$ is called the f-vector of G. The number q is the **maximal dimension** of the graph.

1.3. The **unit sphere** of a vertex v is the sub-graph generated by all its neighbors: $S(x) = (W = \{w \in V, (v, w) \in E\}, \{e = (a, b) \in E, a \in W, b \in W\})$. It has maximal dimension $\leq q-1$ if G has maximal dimension q. The **curvature of a vertex** is v is defined as

$$K(v) = 1 - \sum_{k=0}^{q-1} \frac{(-1)^k f_k(S(v))}{k+1} \; .$$

Theorem: $\chi(G) = \sum_{v \in V} K(v)$.

Proof. Distribute the energy $w(x)$ of each simplex x of dimension k equally to each of its $k+1$ vertices. Then $K(v)$ is the total energy, that v has collected. □

1.4. A different proof is obtained by noting the $\chi(G)$ is a linear combination of valuations $f_k(G)$ which satisfy $f_k(G) = \frac{1}{k+1} \sum_v f_{k-1}(S(v))$ generalizing the **Euler handshake formula** $f_1(G) = \frac{1}{2} \sum_v f_0(S(v)) = \frac{1}{2} \sum_v d(v)$, where $d(v) = f_0(S(v))$ is the **vertex degree** of v.

1.5. To generalize the theorem to simplicial complexes G, let V be the set of 0-dimensional simplices in G. V is naturally identified with $\bigcup_{x \in G} x$. Given $x \in G$, define $U(x) = \{y \in x, x \subset y\}$. The unit sphere is $S(v) = \{x, v \in x, \{v \neq x\}\} = \overline{U(v)} \setminus U(v)$ is a simplicial complex for which $f_k(S(v))$ is defined. Curvature is supported on V.

1.6. The **simplex generating function** $f_G(t) = 1 + f_0(G)t + \cdots + f_d(G)t^{q+1}$ of G satisfies $\chi(G) = 1 - f_G(-1)$. The derivative of a polynomial can be defined without taking limits by using the rule $f' = \sum_{n=0}^q a_n n x^{n-1}$ if $f = \sum_{n=0}^q a_n x^n$. With the **curvature generating function** $K_G(t) = \int_0^t f_G(s)\, ds$, we have $K(v) = K_{S(v)}(-1)$. Gauss-Bonnet is now an elegant recursive formula:

Theorem: $f_G'(t) = \sum_{v \in V} f_{S(v)}(t)$.

Proof. Use the Gauss-Bonnet theorem and check that $K(v) = \int_{-1}^0 f_{S(v)}(t)\, dt$. □

Examples:

1) If G is 1-dimensional, there are no triangles and $\chi(G) = f_0(G) - f_1(G) = |V| - |E|$. The curvature is $K(v) = 1 - \deg(v)/2$, where $\deg(v)$ is the **vertex degree**.
2) **Circular graphs** C_n for example have curvature constant 0.
3) **Star graphs** S_n have curvature $1/2$ on the n leafs and curvature $1 - n/2$ at the branch center. The total curvature of a star graph is 1.
4) **Path graphs** P_n have curvature $1/2$ at the boundary and 0 else. The **cube graph** C has constant curvature $K(v) = 1 - 3/2 = -1/2$ leading to $\chi(C) = -4$. 5) The **dodecahedron graph** D has $K(v) = 1 - 3/2 = -1/2$ and $\chi(D) = -10$.

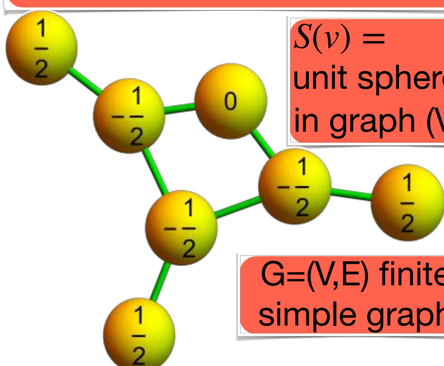
6) A graph is called a **2-manifold** if every unit sphere $S(v)$ is a circular graph of length ≥ 4. The Euler characteristic is then $\chi(G) = |V| - |E| + |F|$. An example is the octahedron O with $|V| = 6$ vertices or the icosahedron I with $|V| = 12$ vertices. The curvature of a 2-dimensional manifold is the **Eberhard formula** $1 - \deg(v)/6$ because $f_0(S(v)) = f_1(S(v)) = \deg(v)$ and $f_0(S(v))/2 - f_1(S(v))/3 = \deg(v)/6$. For the octahedron, the curvature is constant $K(v) = 1 - 4/6 = 1/3$ adding up to $\chi(G) = 2$. For the icosahedron, the curvature is constant $K(v) = 1 - 5/6 = 1/6$ adding up to $\chi(G) = 2$.
7) A complete graph K_n has n vertices and constant $K(v) = 1/n$. One can see this by noting that the curvature must be constant and add up to 1 and that the Euler characteristic of K_n is 1. Note that $f_k(K_n) = \binom{n}{k}$ so that $(1+t)^n = \sum_{k=0}^n f_{k-1}(K_n) t^k$ and for $t = -1$, this is 0 to the left and $-1 + \chi(K_n)$ on the right. For the **tetrahedron** $T = K_4$ for example, the curvature is constant $1/4$. It is not a 2-manifold.
8) The 4-partite graph $K_{2,2,2,2}$ is the **3-dimensional cross polytope**. It is a discrete **3-sphere** or 16-cell. Its f-vector is $f = (f_0, f_1, f_2, f_3) = (|V|, |E|, |F|, |C|) = (8, 24, 32, 16)$, where F is the set of triangles, the 2-dimensional parts of space and C is the set of tetrahedra, the 3-dimensional parts of space. The Euler characteristic is $\chi(G) = 8 - 24 + 32 - 16 = 0$, like for any odd-dimensional manifold. The curvature is constant 0.

Gauss-Bonnet

$$\sum_{x \in V} K(v) = \chi(M)$$

$\chi(G) = \sum_k (-1)^k f_k(G)$ Euler char.

$$K(v) = 1 - \sum_{k=0}^{q-1} (-1)^k \frac{f_k(S(v))}{k+1}$$



$S(v) =$ unit sphere in graph (V,E)

G=(V,E) finite simple graph

$f_k(A) = \#$ of k − simplices in A

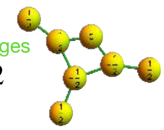
More general:

$$f'_G(t) = \sum_{v \in V} f_{S(v)}(t)$$

simplex generating function
$f(t) = 1 + f_0 t + f_1 t^2 + \ldots + f_q t^{q+1}$

$S(v) = \{w \in V, (v,w) \in E\}$
unit sphere of a vertex v

Example: 7 vertices 7 edges



$f_G(t) = 1 + 7t + 7t^2$

$f'_G(t) = 7 + 14t$

$f(-1) = \chi(G)$

simplex generating functions of S(v) add up to f'(t)

$$\begin{bmatrix} 3t+1 \\ 2t+1 \\ 3t+1 \\ 3t+1 \\ t+1 \\ t+1 \\ t+1 \end{bmatrix}$$ functional curvatures

Origin

Descartes (1623)
De solidorum elementis

$$\alpha(v) = 2\pi(1 - \frac{d(v)}{6})$$

Eberhard (1891)
Morphologie der Polyheder

$$\sum_k (6-k) p_k = 12$$

Heawood (1890)
Discharge in graph coloring

$$6 - d(v)$$

Levitt (1992):
for simplicial complexes

$$K(v)$$

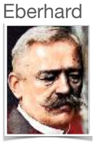
Knill (2012):
rediscover Levitt's curvature graph set-up

$K_2 = 1 - E/6$
$K_4 = 1 - E/6 + F/10$
small dim. manifolds

Knill (2019):
Dehn-Sommerville story

$$f'_G(t) = \sum_{v \in V} f_{S(v)}(t)$$

Eberhard



Heawood

Levitt

Continuum

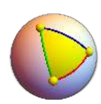
$\alpha, \beta, \gamma =$ flat angles

$\pi = \alpha + \beta + \gamma$

Euclid

$\alpha, \beta, \gamma =$ spherical angles

$\pi + A = \alpha + \beta + \gamma$



Harriot

$\sum_i \kappa_i = 4\pi$

Descartes

$\kappa_i =$ excess angle

$\int_0^L \kappa_g(t)\, dt = 2\pi$

Hopf

$\kappa_g = \dot{x} \times \ddot{x}$

Gauss

$K = \det(A) = \frac{\det(II)}{\det(I)}$ $\kappa_g = (\dot{x} \times n) \cdot \ddot{x}$

$$\int_U K\, dV + \sum_i \int_{C_i} \kappa_g\, dt + \sum_i \kappa_i = 2\pi$$

$\kappa_i =$ complentary angle

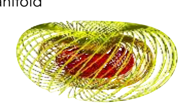
$\int_U K\, dV = 2\pi\chi(M)$

Bonnet

dk Gauss Bonnet Chern integrand
M compact 2k-dim Riemannian manifold

Chern

$$6\int_M K\, dV = \chi(M)$$



Unit 2: Poincare-Hopf

The Poincare-Hopf theorem writes the Euler characteristic of a space as the degree of a divisor, the sum of integer indices attached to points. This works for locally injective scalar function or locally non-circular vector fields.

2.7. Geometry is again a simplicial complex G from a finite simple graph (V, E). A **scalar function** g is a map $g : V \to R$, where R is a totally ordered set, like a subset of $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. We assume that g is a **coloring**, meaning that it is **locally injective**: $g(v) \neq g(w)$ if $(v, w) \in E$. For $v \in V$, the **stable sphere** $S_g^-(v)$ is the sub-graph of the unit sphere $S(v)$, generated by $\{w \in S(v), g(w) < g(v)\}$. Define the **index** $i_g(v) = 1 - \chi(S_g^-(v))$.

Theorem: $\chi(G) = \sum_{v \in V} i_g(v)$.

Proof. Distribute the energy $\omega(x) = (-1)^{\dim(x)}$ of each simplex $x \in G$ to the vertex w in x, where g takes its maximum. Then $i_g(v)$ is the total energy which the vertex v has collected because every k-simplex x in $S_g^-(v)$ gives its energy $\omega(x)$ to v. $\square$

2.8. For the **simplex generating function** $f_G(t) = 1 + f_0(G)t + \cdots + f_d(G)t^{d+1}$, this means:

Theorem: $f_G(t) = 1 + t \sum_{v \in V} f_{S_g^-(v)}(t)$.

Proof. The k-simplex x in $S(v)$ matches up with a $(k+1)$-simplex in G that contains v. This shift of dimension is the reason for the multiplication with t. $\square$

2.9. This formula again gives a recursive computation of $f_G(t)$ and so the cliques of G. It is even more efficient than Gauss-Bonnet because $S_g^-(v)$ is in general smaller than $S(v)$. The index of v is a special value of the generating function of $S_g^{-1}(v)$:

$$i_g(v) = f_{S_g^-(v)}(-1) \ .$$

2.10. A **vector field** is a map $F : G \to V$ with $F(x) \in x$. A **directed graph** with the property that there are no closed loops in each simplex x defines a vector field by $F(x) = \max_{v \in x}(v)$, as the directions on each edge define a total order on x. We call this a **locally non-circular digraph**.

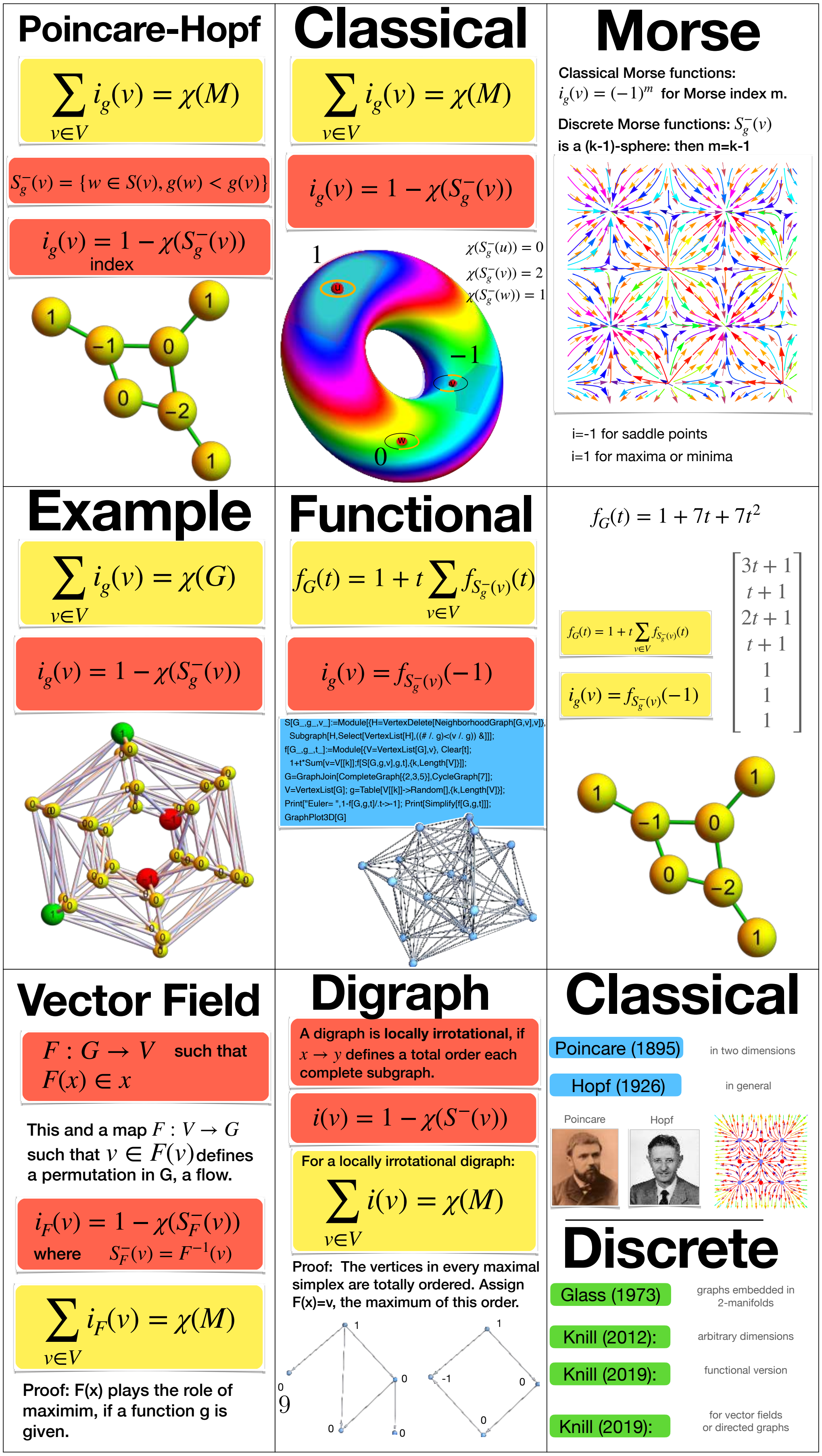
2.11. Define $S_F^-(v) = \{x \in G, F(x) = v\} = F^{-1}(v)$ and $i(v) = 1 - \chi(S_F^-(v))$. We again have $f_G(t) = 1 + t \sum_{v \in V} f_{S_F^-(v)}(t)$.

2.12. An example is the **gradient field** defined by a locally injective function g. In this case $F(x) = v$, where v is the maximum of g on x. The index of such a graph is then $i(v) = 1 - \chi(S^-(v))$, where $S^-(v) = \{w \in S(v), w < v\}$.

Theorem: For a locally non-circular directed graph, $\chi(G) = \sum_{v \in V} i(v)$.

2.13. Define the **symmetric index** $j_g(v) = [i_g(v) + i_{-g}(v)]/2$. By linearity, we still have $\sum_{v \in V} j_g(v) = \chi(G)$.

(1) Let $G = C_n$ be a cycle graph and g a function. Now $i_g(v) = 1$ for local minima and $i_g(v) = -1$ for local maxima. Since minima and maxima alternate, there are the same number of minima and maxima.
(2) If $G = S_n$ is a star graph, and the maximum is at the center c then $i_g(c) = n-1$ and all leaves have $i_g(v) = 1$. The total of all index values is 1. If $g(c)$ is the k'th maximal entry then $i_g(c) = n-k$ and $k-1$ entries that are smaller produce leaf indices 1.
(3) If v is a local minimum of g on V, then $i_g(v) = 1$ because $S(v) = 0$ is the empty graph which has Euler characteristic 0. If v is a local maximum of g on V, then $i_g(v) = 1 - \chi(S(v))$. For a 2-manifold for example where $S(v)$ is a circular graph we have $i_g(v) = 1$ at a maximum too. At a saddle point, where $S(v)$ consists of two disconnected arcs, we have $i_g(v) = -1$.
(4) If G is a complete graph and g is an arbitrary function, then $i_g(v) = 1$ at the minimum and $i_g(v) = 0$ else. The symmetric index is 1/2 on the maximum and 1/2 on the minimum.
(5) If G is a manifold for which $\chi(S(v)) = 2$ for all v like for odd dimensional manifolds, then $\chi_{S^-_{-g}}(v) = 2 - S^-_g$ which can be written as $i_g(v) = -i_{-g}(v)$ so that $j_g(v) = 0$ for all v. Odd dimensional manifolds all have Euler characteristic 0, independent of whether they are orientable.
(6) If G_1 is the Barycentric refinement of G in which the vertices are the simplices of G and two vertices are connected if one is contained in the other, look at the coloring $g(x) = \dim(x)$. Now $S^-_g(x)$ is the boundary complex of the simplex x and $1 - \chi(S^-_g(x)) = \omega(x)$. The Poincaré-Hopf formula now tells that $\chi(G) = \sum_x \omega(x) = \sum_{v \in V(G_1)} i_g(v) = \chi(G_1)$. The Barycentric refinement G_1 has the same Euler characteristic than G.


Poincare-Hopf
$\sum_{v \in V} i_g(v) = \chi(M)$
$S_g^-(v) = \{w \in S(v), g(w) < g(v)\}$
$i_g(v) = 1 - \chi(S_g^-(v))$
index
1
-1
0
1
0
-2
1
Classical
$\sum_{v \in V} i_g(v) = \chi(M)$
$i_g(v) = 1 - \chi(S_g^-(v))$
1
$\chi(S_g^-(u)) = 0$
$\chi(S_g^-(v)) = 2$
$\chi(S_g^-(w)) = 1$
-1
0
Morse
Classical Morse functions:
$i_g(v) = (-1)^m$ for Morse index m.
Discrete Morse functions: $S_g^-(v)$
is a (k-1)-sphere: then m=k-1
i=-1 for saddle points
i=1 for maxima or minima
Example
$\sum_{v \in V} i_g(v) = \chi(G)$
$i_g(v) = 1 - \chi(S_g^-(v))$
Functional
$f_G(t) = 1 + t \sum_{v \in V} f_{S_g^-(v)}(t)$
$i_g(v) = f_{S_g^-(v)}(-1)$
S[G_,g_,v_]:=Module[{H=VertexDelete[NeighborhoodGraph[G,v],v]},
Subgraph[H,Select[VertexList[H],((# /. g)<(v /. g)) &]]];
f[G_,g_,t_]:=Module[{V=VertexList[G],v}, Clear[t];
1+t*Sum[v=V[[k]];f[S[G,g,v],g,t],{k,Length[V]}]];
G=GraphJoin[CompleteGraph[{2,3,5}],CycleGraph[7]];
V=VertexList[G]; g=Table[V[[k]]->Random[],{k,Length[V]}];
Print["Euler= ",1-f[G,g,t]/.t->-1]; Print[Simplify[f[G,g,t]]];
GraphPlot3D[G]
$f_G(t) = 1 + 7t + 7t^2$
$f_G(t) = 1 + t \sum_{v \in V} f_{S_g^-(v)}(t)$
$i_g(v) = f_{S_g^-(v)}(-1)$
$\begin{bmatrix} 3t+1 \\ t+1 \\ 2t+1 \\ t+1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
1
-1
0
1
0
-2
1
Vector Field
$F : G \to V$ such that
$F(x) \in x$
This and a map $F : V \to G$
such that $v \in F(v)$ defines
a permutation in G, a flow.
$i_F(v) = 1 - \chi(S_F^-(v))$
where $S_F^-(v) = F^{-1}(v)$
$\sum_{v \in V} i_F(v) = \chi(M)$
Proof: F(x) plays the role of
maximim, if a function g is
given.
Digraph
A digraph is **locally irrotational,** if
$x \to y$ defines a total order each
complete subgraph.
$i(v) = 1 - \chi(S^-(v))$
For a locally irrotational digraph:
$\sum_{v \in V} i(v) = \chi(M)$
Proof: The vertices in every maximal
simplex are totally ordered. Assign
F(x)=v, the maximum of this order.
1
0
0
0
0
1
-1
0
0
Classical
Poincare (1895)
in two dimensions
Hopf (1926)
in general
Poincare
Hopf
Discrete
Glass (1973)
graphs embedded in
2-manifolds
Knill (2012):
arbitrary dimensions
Knill (2019):
functional version
Knill (2019):
for vector fields
or directed graphs

Unit 3: Index Expectation

Integral geometry links Poincaré-Hopf and Gauss-Bonnet: curvature is the expectation of indices. This is a tool that can link the discrete with the continuum.

3.14. Let G be a simplicial complex from a (V, E) is a finite simple graph and let R be a totally ordered set, like an interval $[a, b]$ or a subset of the integers. Consider $(\Omega, \mathcal{A}, \mathrm{P})$, a **probability space** of locally injective functions from V to R. An example is the set of all possible colorings from V to $\{1, \dots, c\}$, where c is the **chromatic number** and assume that P is the counting measure giving each coloring the same probability. For fixed v, the index $g \to i_g(v)$ is now an integer valued **random variable** on Ω.

3.15. The **index expectation** $K(v) = \mathrm{E}[i_g(v)]$ does not involve g any more. it defines a curvature that depends on the probability space.

Theorem: If $K(v) = \mathrm{E}[i_g(v)]$, then $\sum_{v \in V} K(v) = \chi(G)$.

Proof. Start with Poincaré-Hopf $\sum_{v \in V} i_g(v) = \chi(G)$. Take the expectation of this formula to get $\mathrm{E}[\sum_{v \in V} i_g(v)] = \mathrm{E}[\chi(G)] = \chi(G)$. Fubini (or rather linearity of expectation) allows to write the left hand side as $\sum_{v \in V} \mathrm{E}[i_g(v)] = \sum_{v \in V} K(v)$. $\square$

3.16. Every $v \in V$ defines a random variable $X_v(g) = g(v)$ on $(\Omega, \mathcal{A}, \mathrm{P})$. If these random variables are independent and have all the same uniform distribution on some interval, we say P has the **Lebesgue property**.

Theorem: If P has the Lebesgue property, then K is the Levitt curvature.

Proof. For every k-simplex $x = (x_0, \dots, x_k)$, the probability that a function g has the maximum on x_j is the same as the values of g outside g are independent and so do not factor in. By assumption, the situation on the simplex is now symmetric with respect to any permutation of the vertices in simplices. This means that each vertex gets $1/(k + 1)$ of the energy $\omega(x)$. $\square$

3.17. Assume c is the **chromatic number** of G. Let Ω denote the set of all c colorings. If P has the uniform distribution on this finite set, we say P is the **coloring probability space**.

Theorem: For the coloring probability space, K is the Levitt curvature.

Proof. The same symmetry argument applies: as the probability measure does not change if we permute the coloring space, each point x_j in a simplex has the same probability of being the maximum. $\square$

3.18. The index itself is curvature if P is supported on a single point g in Ω. By changing the P, we can **deform** a geometry. Index expectation allows us to modify the curvature similarly as a Riemannian metric does in the continuum.

3.19. If the graph (V,E) is a triangulation of a compact Riemannian manifold M that is isometrically Nash embedded into an ambient Euclidean space E, then almost all linear functions $g(x) = x \cdot a$ induce Morse functions on M. For a sufficiently fine triangulation G in M, almost all linear functions in the ambient space are locally injective on G. The Poincaré-Hopf indices at critical points match to the classical ones. The index expectation is a curvature that is locally homogeneous. An argument of Weyl implies that it has to be the Gauss-Bonnet-Chern integrand.

3.20. Curvature is compatible with the **Shannon product** $A * B$ of two graphs $A = (V,E), B = (W,F)$ is $(V \times W, Q)$ with edges $Q = \{((a,b),(c,d)), \mathrm{dist}(b,d) \leq 1 \text{ and } \mathrm{dist}(a,c) \leq 1\}$ where dist stands for the **graph distance**. It is also called **strong product**

Theorem: The curvature $K(x,y)$ on $A * B$ is $K(x,y) = K(x)K(y)$.

Proof. (i) First show $i_{G*H,gh}(x,y) = i_{G,g}(x)i_{H,h}(y)$. Proof: For any graphs G, H, we have $(1-\chi(G))(1-\chi(H)) = (1-\chi(G \oplus H))$, where $G \oplus H$ is the graph join. The formula $(1-\chi(S_{G*H,g*h}(x))) = (1-\chi(S_{G,g}(x)))(1-\chi(S_{H,h}(x)))$ follows now from $S_{G*H,g*h}(x,y)$ being homotopic to the join of $S_{G,g}(x)$ and $S_{G,h}(y)$ and that the Euler characteristic is a homotopy invariant. We have $S^-_{G*H}(x,y) = B^-_G(x) * S^-_H(y) \cup S^-_G(x) * B^-_H(y)$, which is homotopic to the join of S^-_H and S^-_G.
(ii) Take the expectation of the relation in (i) and make use of the fact that the random variables $X(g) = i_{G,g}(x)$ and $Y(h) = i_{H,h}(y)$ are independent. The expectation of the product is the product of the expectations. Since $K(x) = \mathrm{E}[X], K(y) = \mathrm{E}[Y]$, we have $K(x,y) = \mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$. $\square$

3.21. Examples: 1) If $A = K_n$ and $B = K_m$, then $A * B = K_{nm}$. The curvature on A is constant $1/n$ the curvature on B is constant $1/m$. The curvature on $A * B$ is constant $1/(nm)$.
2) Assume G is embedded in $\mathbb{R}^2$, meaning $P(v) = (x(v), y(v))$ is given. Let $g(v) = \cos(\theta)x(v) + \sin(\theta)y(v)$ and $\Omega = \{\theta \in [0, 2\pi)\}$. Index expectation defines a curvature that depends on the embedding.
3) Since the symmetric index $[i_g + i_{-g}]/2$ is zero for an odd dimensional manifold any probability measure that is invariant under the involution $g \to -g$ produces a curvature that is constant zero on an odd-dimensional manifold.

Index expectation

(V, E) finite simple graph
$(\Omega, \mathcal{A}, P)$ probabiliy space of coloring functions on V

$$K(v) = \mathrm{E}[i_g(v)]$$

defines a curvature that satisfies Gauss-Bonnet:

$$\sum_{v \in V} K(v) = \chi(G)$$

Proof: take expectation of the Poincare-Hopf formula:

$$\sum_{v \in V} i_g(v) = \chi(G)$$

$$\mathrm{E}[\sum_{v \in V} i_g(v)] = \chi(G)$$

and apply Fubini

$$\sum_{v \in V} \mathrm{E}[i_g(v)] = \chi(G)$$

Continuum

(M,g) compact Riemannian manifold
$(\Omega, \mathcal{A}, P)$ probabiliy space of Morse functions on M

$$K(v) = \mathrm{E}[i_g(v)]$$

satisfies Gauss-Bonnet. For nice probability spaces, it is a smooth function on M

$$\int_M K(v)\, dV = \chi(G)$$

Rewrite Poincare -Hopf:

$$\sum_{v \in M} i_g(v) = \chi(M)$$

as

$$\int_M i_g(v) = \chi(M)$$

where now,

i_g divisor (a signed point measure).

Now apply expectation and Fubini.

Gauss Bonnet Chern

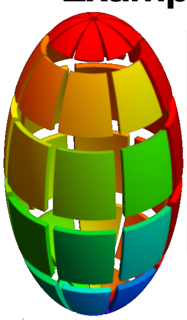
(M,g) Riemannian, compact even dimensional

$$\int_M \sum_\pi \frac{\prod_{k=1}^d K_{\pi(2k-1),\pi(2k)}}{d!(4\pi)^d} dV = \chi(M)$$

Weyl and Gilkey:
The GBC integrand is the unique local frame invariant curvature.

The index expectation coming from a Nash embedding into an Euclidean space must be the Gauss-Bonnet-Chern curvature!

Example:



$$r(s,t,u,v) = \begin{bmatrix} 2\cos(t) \\ \cos(s)\sin(t) \\ \sin(s)\sin(t)\cos(u) \\ \sin(s)\sin(t)\sin(u)\cos(v) \\ \sin(s)\sin(t)\sin(u)\sin(v) \end{bmatrix}$$

4 dim ellipsoid

$$M \subset \mathbb{R}^5$$

$$K(x) = -\frac{128}{(5 - 3\cos(2t))^2(3\cos(2t) - 5)}$$

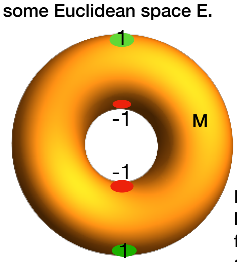
$$dV = \frac{\sqrt{\sin^4(s)\sin^6(t)(-(3\cos(2t) - 5))\sin^2(u)}}{\sqrt{2}}$$

Gauss-Bonnet-Chern

$$\int_M K(x)dV(x) = 2$$

Discrete-Continuum

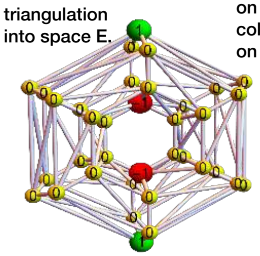
Nash embed manifold M into some Euclidean space E.



g(x,y,z) =z here

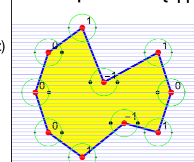
Most height functions are Morse on M and colorings on G.

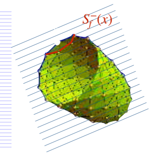
Embed the triangulation into space E.



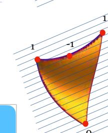
Nash-embed

Use rotational symmetric measure on sphere S = {v| |v|=1}

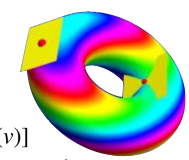




Nash embed M into Euclidean space E. Use linear functions



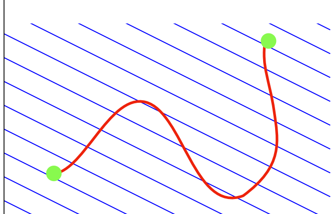
$$g_v(x) = x \cdot v$$



$K(v) = \mathrm{E}[i_g(v)]$

is GBChern integrand

Integral Geometry



Prototype result of Crofton: The length of a curve is the proportional to the expected number $\mathrm{E}[i_\alpha]$ of intersections

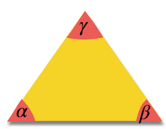
$$i_\alpha = |\{g_\alpha(x) \in \mathbb{Z}\}|$$

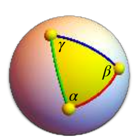
with an equidistant parallel grid. See Buffon needle.

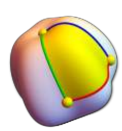
Morgan Crofton (1826-1915)

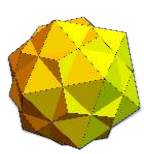
Curvature Flavors

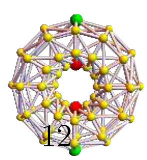
Any curvature in differential geometry can be written integral geometrically!











1827 Gauss

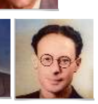
1848 Bonnet

1888 von Dyck

1940 Allendoerfer/ Fenchel

1927 Hopf

1943 Allendoerfer -Weil



1944 Chern

Unit 4: Euler's Gem

A q-sphere G is a q-manifold which when punctured becomes contractible. Euler's Gem formula tells that Euler characteristic is $\chi(G) = 1 + (-1)^q$. Platonic q-spheres are inductively defined as q-spheres for which all unit spheres are isomorphic and equal to some Platonic $(q-1)$-sphere. They can be classified.

4.22. A graph (V, E) is **contractible** if there exists $v \in V$ such that $S(v)$ and $G \setminus v$ are both contractible. The 1-point graph $1 = K_1$ is contractible. The (-1)-sphere 0 is not contractible.

4.23. A graph G is called a **q-manifold** if all unit spheres $S(v)$ are $(q-1)$-spheres. A ***q*-sphere** is a q-manifold for which there is a v such that $G \setminus v$ is contractible.

4.24. Every contractible graph has $\chi(G) = 1$. One can see this by induction: $\chi(G) = \chi(G \setminus v) + \chi(S(v)) - \chi(\{v\}) = 1 + 1 - 1 = 1$. If A, B are graphs, $A \cap B$ carries the intersection of the simplicial complexes of A and B, then the **valuation formula** for simplicial complexes holds also for graphs $\chi(A) + \chi(B) - \chi(A \cap B) = \chi(A \cup B)$. $A \cup B$ carries the intersection of the simplicial complexes of A and B. The Euler Gem formula is:

Theorem: $\chi(G) = 1 + (-1)^q$ for a q-sphere.

Proof. (i) In every graph, every unit ball $B(v)$ is contractible. It follows that $\chi(B(v)) = 1$. (ii) for a q-sphere, every unit sphere $S(v)$ is a $(q-1)$-sphere and so has by induction the Euler characteristic $1 + (-1)^{q-1}) = 1 - (-1)^q$.
(iii) Now use $\chi(G) = \chi(G \setminus v) + \chi(B(v)) - \chi(S(v)) = 1 + 1 - (1 + (-1)^{q-1}) = 1 + (-1)^q$. We used the valuation formula. □

4.25. The **join** of two graphs $A \oplus B$ has vertex set $V(A) \cup V(B)$ and edge set $E(A) \cup E(B) \cup \{(a, b), a \in V(A), b \in V(B)\}$. It is the dual of disjoint union $+$ because $A \oplus B = \overline{\overline{A} + \overline{B}}$. With this operation and the zero element 0 we have a second monoid, dual and isomorphic to the monoid with disjoint union. Spheres form a sub-monoid of the join monoid.

Theorem: A is a k-sphere, B is a l-sphere $\Rightarrow$ $A \oplus B$ is a $(k + l + 1)$-sphere.

Proof. Use induction with respect to $k + l = n$. For $n = 0$ meaning $k = l = 0$, the join is a 1-sphere. Assume we have proven it for all cases with $k + k \leq n - 1$ and assume $k + l = n$. By definition every unit sphere $S_A(v)$ is a $(k-1)$-sphere and every unit sphere $S_B(w)$ is a $(l-1)$-sphere. For $v \in V(A)$, the graph $S_{A \oplus B}(v) = S_A(v) \oplus B$ is a $l - 1 + k + 1$-sphere by induction. For $w \in V(B)$, the graph $S_{A \oplus B}(w) = A \oplus S_B(w)$ is a $l + k - 1 + 1$-sphere by induction. This shows that $A \oplus B$ is a $l + k + 1$ manifold. Now use that if A is contractible and B is arbitrary than $A + B$ is contractible. This shows that for $v \in V(A)$, the graph $A \oplus B \setminus v = (A \setminus v) \oplus B$ is contractible. Similarly, for $w \in V(B)$, the graph $A \oplus B \setminus w = A \oplus (B \setminus w)$ is contractible. $\square$

4.26. Inductively, a q-sphere is declared to be **Platonic**, if all unit spheres are Platonic $(q-1)$- spheres that are all isomorphic. 0 is Platonic. There is a unique Platonic sphere in all dimensions except dimensions 1,2,3. There are infinitely many 1-spheres C_n and they are all Platonic. For $d = 2$,there is the **octahedron** and **icosahedron**. For $d = 3$, there is the **600 cell** and the **16 cell**. After that there are only **cross polytopes**, the 2^{q+1} cell.

Theorem: There is a unique Platonic sphere for $q > 3$.

Proof. $q = -1, 0, 1$ are clear. For $q = 2$, the curvature $K(x) = 1 - f_0/2 + f_1/3 - f_2/3$ is constant, adding up to 2. It is either 1/3 or 1/6. For $d = 3$, where each $S(x)$ must be either the octahedron or icosahedron, G is the 16 cell or 600 cell. For $d = 4$, by Gauss-Bonnet, $K(x)$ add up to 2 and be of the form $L/12$ for some integer L. For $L = 1$, there exists the 4-dimensional cross polytope with f-vector $(10, 40, 80, 80, 32)$. There is no 4-sphere, for which $S(x)$ is the 600-cell. As the f-vector of it is $(120, 720, 1200, 600)$, we would get $K(x) = 1 - 120/2 + 720/3 - 1200/4 + 600/5 = 1$. Gauss-Bonnet would give $|V| = 2$ and $\dim(G) \leq 1$. Inductively there is now only one Platonic q-sphere for $q > 3$. $\square$

4.27. In classifications of **Platonic solids**, **dual polytopes** are usually included. Since they are triangle free, they are not q-manifolds. Classics also add the (q+1)-simplices because their boundary simplicial complex are $(q-1)$-spheres as simplicial complexes. The 24-cell U, the units of the Hurwitz quaternions is the only additional polytop. One could prove this by extending Gauss-Bonnet to combinatorial cell-complexes: inductively add k-cells to an already existing $(k-1)$ sphere. The notion of q-sphere, q-manifold, contractibility and Platonic sphere can be generalized to such discrete complexes similarly as CW complexes in the continuum. After extending Gauss-Bonnet in the same way, we build up the Platonic spheres inductively by dimension. U does not occur as a unit sphere of any Platonic 4-complex so that the miracle in dimension $q = 3$ ends. Traditionally, the continuum is involved: regular q-polytopes are boundaries of convex regions in $\mathbb{R}^{q+1}$.

Euler's Gem

(V, E) finite simple graph

Inductive definitions:

G is contractible: exists v, such that S(v) and G-v are contractible. 1 is contractible.

G is q-manifold: all S(v) are (q-1)-spheres. **G is q-sphere:** exists v such that G-v is contractible. 0 is the (-1)-sphere.

Euler's Gem

If G is a q-sphere, then
$$\chi(G) = 1 + (-1)^q$$

Descartes Euler Schläfli

Lemmas

A

Every B(v) is contractible.

Proof: Induction. Take $w \in V(B)\setminus v$. Then B(w) is smaller and so contractible. S(w) is unit ball in a smaller graph.

B

G contractible implies $\chi(G) = 1$

Proof: induction with n=|V|. Pick v such that $G\setminus v$ is contractible
$\chi(G) = \chi(G\setminus v) + \chi(B(v)) - \chi(S(v)) = 1 + 1 - 1$

C

$\chi(A \cup B) + \chi(A \cap B) = \chi(A) + \chi(B)$

Proof: Use linearity and and check it for valuations $A \to f_k(A)$

Proof of Gem

If G is a q-sphere, then
$$\chi(G) = 1 + (-1)^q$$

Proof: Induction. Check q=0.

Pick v such that G-v is contractible.

Use C

$\chi(G) = \chi(G\setminus v) + \chi(B(v)) - \chi(S(v))$

Fill in the inductive assumption.

$\chi(S(v)) = 1 - (-1)^q$

Use

$\chi(G\setminus v) = 1$ B

and

$\chi(B(v)) = 1$ A B

This gives $\chi(G) = 1 + 1 - (1 + (-1)^{(q-1)})$

Small q

If G is a q-sphere, then
$$\chi(G) = 1 + (-1)^q$$

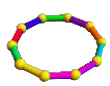
q=-1 χ(G)=0

q=0

|V|=2

χ(G)=2

q=1



|V|=10
|E|=10

χ(G)=10-10=0

q=2

|V|=32 |F|=60
|E|=90

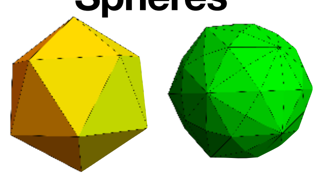
χ(G)=32-90+60=2

q=3

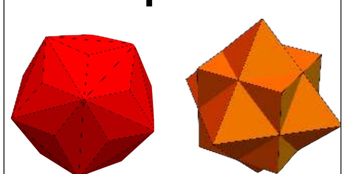
|V|=8 |F|=32
|E|=24 |C|=16

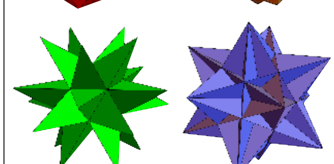
χ(G)=8-24+32-16=0

Spheres

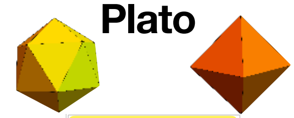


No Spheres

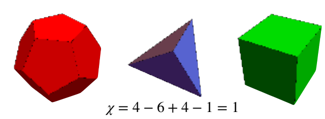




Plato



$V - E + F = 2$

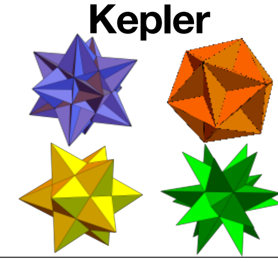


$\chi = 4 - 6 + 4 - 1 = 1$
maxdim=3

$\chi = 20 - 30 = -10$
maxdim=1

$\chi = 8 - 12 = -4$
maxdim=1

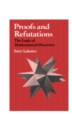
Kepler



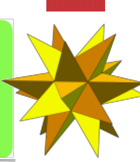
Epistemology

Imre Lakatos 1922-1974

"Monster Barring"
"Monster Adjustment"
"Monster Handling"

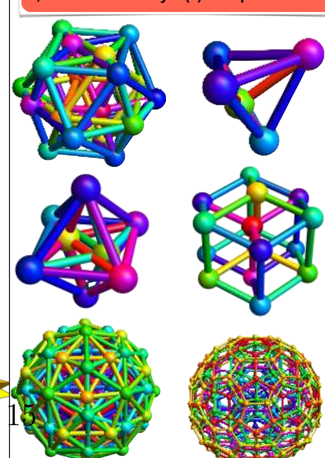


Better: avoid monsters by making precise, unambiguous and simple definitions!



Platonic Spheres

G is platonic q-sphere if it is a q-sphere and if there is a Platonic (q-1) sphere A, such that every S(v) is equal to A.



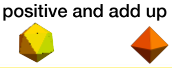
Classification

∞ many Platonic 1-spheres

Proof: every connected 1-manifold is a circular graph C(n), n>3.

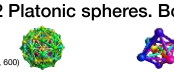
2 Platonic 2-spheres.

Proof: Gauss-Bonnet. Curvature must be positive and add up to 2.



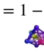
2 Platonic 3-spheres.

Proof: Unit sphere must be one of the two q=2 Platonic spheres. Both work:



1 Platonic 4-sphere.

Proof: Gauss-Bonnet. Curvature of a 4-manifold is $K = 1 - E/6 + F/10$

 K=1  K=1/5

Being down to 1, there is only one Platonic sphere for all q>3.

Unit 5: Euler-Poincaré

A simplicial complex defines an exterior derivative d on the Hilbert space of forms. The kernels of $H = (d + d^*)^2$ restricted to k-forms is the k'th Betti number. The alternating sum of the b_k is the Euler characteristic.

5.28. Let G be a finite abstract simplicial complex. It can be the Whitney complex of a graph for example. If G has cardinality n, the vectors in the Hilbert space $\Lambda = l^2(G) \sim \mathbb{R}^n$ are **discrete differential forms** or simply **forms**. Functions on $G_k = \{x \in G, \dim(x) = k\}$ are called k**-forms**. There is a natural decomposition $\Lambda = \oplus_{k=0}^q \Lambda_k$ and linear maps on Λ that preserve Λ_k are block matrices if we assume the elements in G to be ordered according to dimension.

5.29. For $y \subset x$ with $\dim(y) = \dim(x) - 1$, define $d(x, y) = \mathrm{sign}(x, y)$, where $\mathrm{sign}(x, y)$ is the signature of the permutation that maps (vy) to x, where $v = x \setminus y$. For example, $\mathrm{sign}((1, 2, 3, 4), (1, 2, 4)) = \mathrm{sign}((1, 2, 3, 4), (3, 1, 2, 4)) = 1$. The matrix d is nilpotent $d^2 = 0$ and is called the **exterior derivative**. It maps k-forms to $(k + 1)$-forms. The symmetric matrix $D = d + d^*$ is the **Dirac operator**. Its square $H = D^2 = dd^* + d^*d$ is the **Hodge operator**.

5.30. H decomposes into blocks $\oplus_{k=0}^q H_k$ because both dd^* and d^*d map Λ_k into itself. The k**'th cohomology group** is defined as the space of k**-harmonic forms** $\ker(H_k)$. Its dimension is the k'th **Betti number** b_k of G. If I_k are the identity matrices on $l^2(G_k)$, then $f_k = \mathrm{tr}(I_k) = |G_k|$ is the number of k-simplices and $\chi(G) = \sum_{k=0}^q (-1)^k f_k$. Similarly as the f-vector $(f_0, f_1, \dots, f_q)$ encodes cardinalities, the **Betti-vector** $(b_0, b_1 \dots, b_q)$ encodes dimensions of kernels.

5.31. The **McKean-Singer symmetry** is a spectral symmetry telling that non-zero eigenvalues on even forms correspond bijectively to non-zero eigenvalues on odd forms. It can be encoded as:

Theorem: $\mathrm{str}(e^{-tH}) = \chi(G)$.

Proof. If u is an eigenvector to a non-zero eigenvalue $Hu = \lambda u$, then $v = Du$ is an eigenvector to the same eigenvalue $Hv = \lambda v$. But if u is an even dimensional form, then v is an odd dimensional form. We have now an isomorphism between even and odd parts of the image of H. It follows that the super trace of H^k is zero for $k > 0$. As $\mathrm{str}(H^0) = \mathrm{str}(1) = \chi(G)$, when we apply the super trace to $e^{-tH} = \sum_{k=0}^\infty t^k H^k/k!$ this number is independent of t and equal to the super trace of H^0 which is $\chi(G)$. $\square$

5.32. The following result is the **Poincaré-Hopf theorem**.

Theorem: $\chi(G) = \sum_{k=0}^q (-1)^k f_k = \sum_{k=0}^q (-1)^k b_k$.

Proof. Use the heat flow e^{-tH} and the McKean-Singer symmetry. For $t = 0$, the super trace of e^{-tH} is the left hand side. For $t = \infty$, we get the right hand side. Since the super trace does not change, the left and right hand side agree. □

5.33. The **Hodge decomposition** tells that any k-form g can be written as a sum of an exact, a co-exact and a harmonic k-form: $g = df + d^*h + k$ and that harmonic forms are the intersection of the kernel of d and d^*.

Theorem: $\mathrm{im}(H) = \mathrm{im}(d) + \mathrm{im}(d^*)$, $\ker(H) = \ker(d) \cap \ker(d^*)$.

Proof. The operator $H : \Lambda \to \Lambda$ is symmetric so that image and kernel are perpendicular. The image of H splits into two orthogonal components $\mathrm{im}(d)$ and $\mathrm{im}(d^*)$. That $f = dg$ and $h = d^*k$ are orthogonal is seen from $\langle dg, d^*k\rangle = \langle ddg, k\rangle = 0$.
To the second statement: the kernel of H is contained both in the kernel of d and the kernel of $d*$ because $Hf = 0$ implies $0 = \langle f, Hf\rangle = \langle d^*f, d*f\rangle + \langle df, df\rangle = |d^*f|^2 + |df|^2$, so that both $d^*f = 0$ and $df = 0$. On the other hand, if $df = 0$ and $d^*f = 0$, then $Hf = dd^*f + d^*df = 0$. □

5.34. Let $d_k : \Lambda_k \to \Lambda_{k+1}$ be the restriction of d to Λ_k. If the kernel $\ker(d_k)$ of dimension z_k and the range $\mathrm{im}(d)$ has dimension r_k, the rank-nullity theorem in linear algebra tells $\dim(\ker(d_k)) + \dim(\mathrm{im}(d_k)) = f_k$, we get $z_k = f_k - r_k$.

Theorem: $b_k = \dim(\ker(d_k))/\mathrm{im}(d_{k-1})$ and so $b_k = z_k - r_{k-1}$.

Proof. $\mathrm{im}(d_{k-1})$ is contained in $\ker(d_k)$. The $n \times f_k$ matrix D_k containing the blocks d^*_{k-1} and d_k satisfies $D^*_k D_k = H_k$. Now use that $\mathrm{im}(d_{k-1})$ is the orthogonal complement of $\ker(d^*_{k-1})$ and row reduce F_k to see that b_k, the number of redundant columns in F_k are the columns, where no leading 1 for d^*_{k-1} and no leading 1 for d_k appears. □

5.35. Adding $z_k = f_k - r_k$ and $b_k = z_k - r_{k-1}$ gives $f_k - b_k = r_{k-1} + r_k$. Taking the alternating sum on both sides using $r_{-1} = 0$ and $r_k = 0$ for $k > q$, telescopes $\sum_{k=0}^q (-1)^k (f_k - b_k) = \sum_{k=0}^\infty (-1)^k (r_{k-1} + r_k) = 0$. This is the traditional proof of the Euler-Poincaré theorem.

5.36. Examples. 1) b_0 is the number of components, b_1 the number 1 dim holes. 2) For connected orientable surfaces, we have $b_0 = b_2 = 1$ so that $\chi(G) = |V| - |E| + |F| = 2 - b_1 = 2 - 2g$ where g is the genus of the surface. 3) For connected regions in the plane we have $|V| - |E| + |F| = 1 - g = b_0 - b_1$. 4) A contractible graph has $b_0 = 1$ and $b_k = 0$ for all $k > 0$. 5) For a q-sphere $b_0 = b_q = 1$ and $b_k = 0$ for all $0 < k < q$.

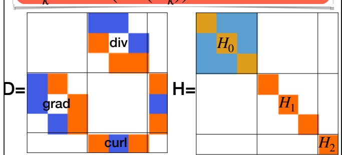
Euler-Poincaré

G: finite abstract simplicial complex.

$\Lambda = l^2(G)$ space of differential forms. Linear space of dim = |G|.

If $y \subset x$, dim(y)=dim(x)-1, define d(x,y) = sign(x,y). Else d(x,y)=0. d is called the exterior derivative.

$D = d + d^*$ Dirac operator
$H = D^2 = dd^* + d^*d$ Hodge
$H = \oplus_{k=0}^q H_k$ block decomposition
$b_k = \dim(\ker(H_k))$ Betti numbers



Euler-Poincaré

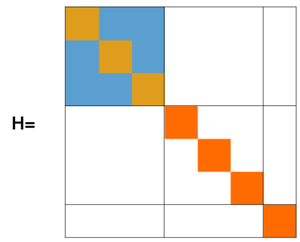
$$\chi(G) = \sum_{k=0}^q (-1)^k b_k$$

McKean-Singer

$\mathrm{str}(H) = \sum_{k=0}^q (-1)^k \mathrm{tr}(H_k)$ super trace

$$\mathrm{str}(e^{-tH}) = \chi(G)$$

$G = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$



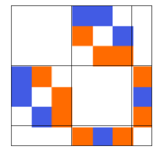
$\sigma(H_0) = \{0,3,3\}$ $\quad H_0 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$\sigma(H_1) = \{3,3,3\}$ $\quad H_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\sigma(H_2) = \{3\}$ $\quad H_2 = [3]$

Proof of MacKean-Singer

$\Lambda = \ker(H) \oplus \mathrm{im}(H)$
$\mathrm{im}(H) = \mathrm{im}(d) \oplus \mathrm{im}(d^*)$
$\Lambda = \Lambda_{\mathrm{even}} \oplus \Lambda_{\mathrm{odd}}$



$D : \Lambda_{\mathrm{even}} \to \Lambda_{\mathrm{odd}}$

defines an isomorphism from $\mathrm{im}(H)_{even} \to \mathrm{im}(H)_{odd}$

$\mathrm{str}(L^n) = 0, \forall n > 0$

$$\mathrm{str}(e^{-tH}) = \mathrm{str}(1 + tH + \frac{H^2}{2}t^2 + \dots) = \mathrm{str}(1) = \chi(G)$$

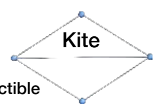
QED

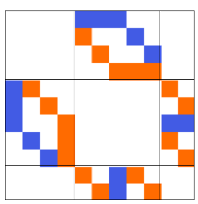
$G = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\}\}$

$G = K_{1,2,1}$

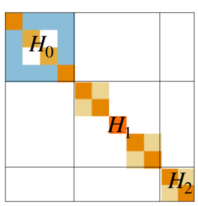
b=(1,0,0)

(b_0, b_1, b_2) contractible





$\sigma(D) = \{-2, -2, 2, 2, 2, 2, 0, -\sqrt{2}, -\sqrt{2}, \sqrt{2}, \sqrt{2}\}$



$\sigma(H_0) = \{4,4,2,0\}$

$\sigma(H_1) = \{4,4,4,2,2\}$

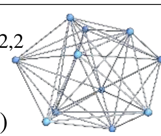
$\sigma(H_2) = \{4,2\}$

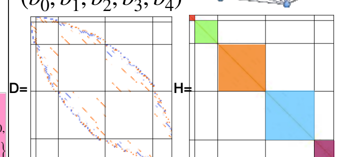
$G = K_5 \oplus K_{2,2,2}$

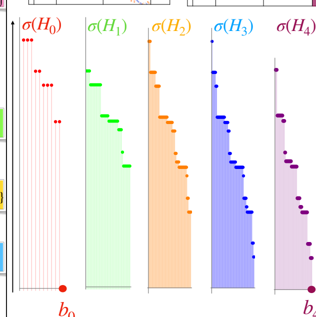
4-sphere

b=(1,0,0,0,1)

$(b_0, b_1, b_2, b_3, b_4)$







Proof of Euler Poincaré

McKean-Singer showed that

$$\mathrm{str}(e^{-tH}) = \mathrm{str}(1 + tH + \frac{H^2}{2}t^2 + \dots)$$

is invariant of t. For t=0 it is

$$\mathrm{str}(1) = \sum_{k=0}^q (-1)^k f_k = \chi(G)$$

$H = D^2$ has non-negative spectrum.

For every eigenvector f of H with positive eigenvalue λ, we have

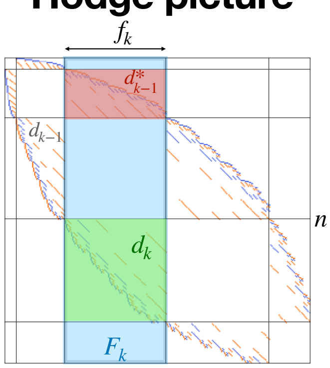
$$e^{-tH} f = e^{-t\lambda} f \to 0$$

Therefore $P = \lim_{t \to \infty} e^{-tH}$ is the projection on ker(H). But

$$\mathrm{str}(P) = \sum_{k=0}^q (-1)^k b_k$$

QED

Hodge picture



$F_k^* F_k = H_k \quad \ker(F_k) = \ker(H_k)$
$= \ker(d_k) \cap \ker(d^*_{k-1})$

$\mathrm{im}(d_{k-1}) \subset \ker(d_k)$

$\ker(d^*_{k-1})^\perp \subset \ker(d_k)$

$\dim(\ker(d_k)/\mathrm{im}(d_{k-1})) = \dim(\ker(H_k))$

Calabi-Yau example

G=K3 complex surface

$x^4 + y^4 + z^4 + w^4 = 0 \subset \mathbb{CP}^3$

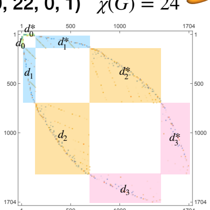
Calabi-Yau manifold of dimension 2
~ smooth real 4 manifold

f=(16, 120, 560, 720, 288)

b=(1, 0, 22, 0, 1) $\chi(G) = 24$

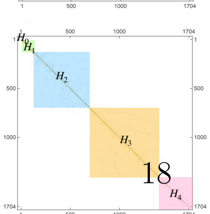
D=

Dirac



H=

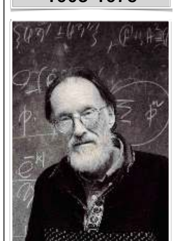
Hodge



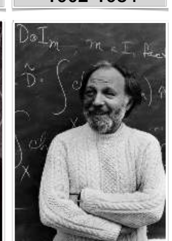
1704x1704 matrices

William Hodge
1903-1975

Paul Dirac
1902-1984



Henry McKean
1930-2024



Isadore Singer
1924-2021

Unit 6: Unimodularity

A simplicial complex G defines a connection matrix L that is unimodular. The inverse matrix g of L is explicitly given in the form of the Green-Star formula $g(x,y) = \omega(x)\omega(y)\chi(U(x) \cap U(y))$. The total potential $\sum_{x,y} g(x,y)$ is $\chi(G)$.

6.37. If G is a simplicial complex, the **stars** $U(x) = \{y \in G, x \subset y\}$ are open sets, while the **cores** $C(x)\{y \in G, y \subset x\}$ are closed. For $A \subset G$, the **Fermi characteristic** $\phi(G) = \prod_{x \in G} \omega(x)$ of is a multiplicative sibling of the additive Euler characteristic $\chi(A) = \sum_{x \in A} \omega(x)$, where $\omega(x) = (-1)^{\dim(x)}$. Define the **connection matrix** $L(x,y) = \chi(C(x) \cap C(y))$. The entries are 1 if $x \cap y \neq \emptyset$ and 0 else. Define also $\psi(G) = \det(L(G))$. The **unimodularity theorem** is:

Theorem: $\det(L) = \phi(G) \in \{-1,1\}$.

Proof. The simplicial complex G can be built up recursively from a smaller complex H by adjoining a new k-simplex x to an already existing $(k-1)$-sphere A in H that is not yet the boundary of a ball in H. This enlargement $H \to G = H \cup_A \{x\}$ changes the Fermi characteristic $\phi(G)$ to $\phi(G) = \phi(G)\omega(x)$. A path interpretation of the determinant verifies that this also changes the determinant $\psi(G) = \det(L)$ by a factor $\omega(x)$. It is the content of the multiplicative Poincaré-Hopf analog below. □

6.38. Poincaré-Hopf is the additive statement $\chi(G \cup_A \{x\}) = \chi(G) + (1 - \chi(A))$. It used the inclusion-exclusion formula and $\chi(B(\{x\})) = 1$. Multiplicatively, we have:

Theorem: $\psi(G \cup_A \{x\}) = \psi(G)(1 - \chi(A))$.

Proof. $Y : A \to (\psi(G \cup_A x) - \psi(G))$ is a valuation that satisfies $\psi(G \cup_A x) = 0$ if A is a complete complex so that $Y(A) = -\psi(G)$ if A is a complete complex. The scaled valuation $-Y(A)/\psi(G)$ takes the value 1 for complete sub-complexes. This characterizes Euler characteristic: we have $-Y(A)/\psi(G) = \chi(A)$ meaning $Y(A) = -\chi(A)\psi(G)$ But $(\psi(G \cup_A x) - \psi(G)) = -\chi(A)\psi(G)$ is equivalent to $\psi(G \cup_A \{x\}) = \psi(G)(1 - \chi(A))$. □

6.39. Let $g(x,y) = \omega(x)\omega(y)\chi(U(x) \cap U(y))$ be the **Green function**. It is like L a $n \times n$ matrix. It invokes the **stars** $U(x), U(y)$ of x and y. The next result is the **Green-star formula**:

Theorem: $L^{-1} = g$.

Proof. Write the matrix entries of L and g are linear expressions in $\omega(x)$:

$$L(u,v) = \sum_{x \in G, x \subset u \cap v} \omega(x) ,$$

$$g(v,w) = \sum_{y \in G, v \cup w \subset y} \omega(v)\omega(w)\omega(y) \; .$$

Now multiply the two matrices $\sum_v L(u,v)g(v,w)$. This is a sum over x,y,v. Distinguish two cases. If $u=w$, we divide the sum up into two parts, one part where $x=y$ which implies $u=w,x,y=v$ and giving 1 and a part where $x \neq y$, where the sum over v is zero. The diagonal entries are 1.
In the second case $u \neq w$ the situation is similar, but $x=y$ is no more possible. We end up with 0. □

6.40. If $g(x,y)$ is interpreted as the **potential energy** between the simplices x and y, the number $E(G) = \sum_{x,y \in G} g(x,y)$ is the **total energy**. The **energy theorem** is:

Theorem: $\sum_{x,y \in G} g(x,y) = \chi(G)$

Proof. We have $g(x,x) = \chi(U(x)) = 1 - \chi(S(x))$. Call it $i(x)$. By the Cramer formula, $g(x,x)$ is $\psi(G \setminus x)/\psi(G)$ by the multiplicative Poincaré-Hopf theorem this is $1-\chi(S(x))$. Now think of G is the vertex set of a graph in which two vertices are connected if one is contained in the other. Take the functional $\dim(x)$. Hyperbolicity gives $S(x) = S_f^-(x) \oplus S_f^+(x)$ so that $i_f(x)i_{-f}(x) = i(x)$. Since $i_f(x) = \omega(x)$ and $i_{-f}(x) = 1-\chi(U(x))$ $k(x) = \omega(x)(1-\chi(S(x)))$ is also a curvature adding up to $\chi(G)$. This implies $\sum_x \omega(x)\chi(S(x)) = 0$ which can be read as a McKean-Singer $\mathrm{str}(L^k) = \chi(G)$ for $k=-1,0,1$. Introduce the potential $V(x) = \sum_y g(x,y)$. It satisfies $V(x) = \omega(x)g(x,x) = k(x)$. Gauss-Bonnet $\sum_x k(x) = \chi(G)$ now proves the theorem. □

6.41. Examples:
1) $\det(L(K_1)) = 1$ but $\det(L(K_k)) = -1$ for $k > 1$.
2) The diamond complex $G = \{\, (1),(2),\ (3),(4),\ (1,2),(1,3),\ (2,3),(2,4),\ (3,4),$ $(1,2,3),(2,3,4)\}$ has f-vector $f = -(4,5,2)$, Euler characteristic $\chi(G) = 4-5+2 = 1$ and Betti vector $b = (1,0,0)$. Its Fermi characteristic is $(+1)^4(-1)^5(+1)^2 = -1$.

Unimodularity

G: finite abstract simplicial complex.

$C(x) = \{y \in G, y \subset x\}$ closed set
$U(x) = \{y \in G, x \subset y\}$ open set

$L(x,y) = \chi(C(x) \cap C(y))$ connection matrix
$g(x,y) = \omega(x)\omega(y)\chi(U(x) \cap U(y))$ Green function

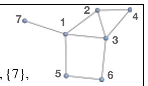
$D = d + d^*$ was based on incidence
L is based on intersection
D was never invertible L always is

$\det(L) = \prod_{x \in G} \omega(x)$ — 1 **Unimodularity**

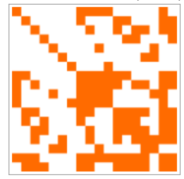
$L^{-1} = g$ — 2 **Green Star**

$\sum_{x,y} g(x,y) = \chi(G)$ — 3 **Energy Theorem**

Example

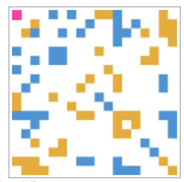


$G = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},$
$\{1,2\},\{1,3\},\{1,5\},\{1,7\},\{2,3\},\{2,4\},\{3,4\},\{3,6\},\{5,6\},$
$\{1,2,3\},\{2,3,4\}\}$ f=(7, 9, 2) b=(1,1,0)

L= 

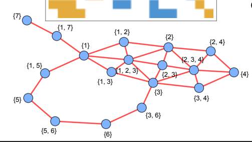
entries of L are
1
0

det(L)=-1

g= 

entries of g are
-2
1
-1
0

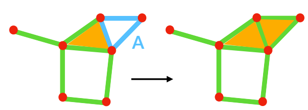
det(g)=-1



Proof of 1

$\det(L) = \prod_{x \in G} \omega(x)$

Build up complex inductively by attaching k-simplex x to (k-1) sphere.



H $G = H \cup_A \{x\}$

Right hand side: multiply by $\omega(x)$

Left hand side: define $\psi(H) = \det(L(H))$

$Y : A \to (\psi(G \cup_A x) - \psi(G))$

is a valuation satisfying Y(A)=-$\psi(A)$ for complete A. Since $-\frac{Y(A)}{\psi(G)} = 1$ for complete A,it must be $\chi(A)$. Therefore,

$$\psi(G \cup_A \{x\}) = \psi(G)(1 - \chi(A))$$
$$= \psi(G)\omega(x)$$

QED

Proof of 2

$L^{-1} = g$

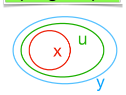
$$L(u,v) = \sum_{x, x \subset u \cap v} h(x)$$

$$g(v,w) = \sum_{y, v \cup w \subset y} \omega(v)\omega(w)h(y)$$

$$\sum_v L(u,v)g(v,w) = \sum_v \sum_{x, x \subset u \cap v} \sum_{y, v \cup w \subset y} \omega(v)\omega(w)h(x)h(y)$$

Case u=w: (diagonal)

$$\sum_v [\sum_{x \subset u \cap v, v \cup w \subset y} h(x)\omega(v)\omega(x)h(y)]$$



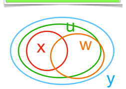
$= h(u)^2 = 1$

(only x=y=v=u produces a non-zero situation)

Case $u \neq w$ (off-diagonal)

$$\sum_v [\sum_{x \subset u \cap v, v \cup w \subset y} h(x)\omega(v)\omega(w)h(y)]$$

$= 0$



(as x=y is now not possible and terms cancel)

QED

Proof of 3

$\sum_{x,y} g(x,y) = \chi(G)$

$g(x,x) = \chi(U(x)) = 1 - \chi(S(x))$

Proof: Cramer gives

$$g(x,x) = \psi(G \setminus x)/\psi(G)$$
$$= (1 - \chi(S(x)))$$

We call the later multiplicative Poincare-Hopf

Hyperbolicity: $S_f^-(x) \oplus S_f^+(x) = S(x)$

So: $i_f(x) i_{-f}(x) = (1 - \chi(U(x)))$

Take coloring f=dim, where $i_f(x) = \omega(x)$

$$k(x) = \omega(x)(1 - \chi(S(x)))$$

is $i_{-f}(x)$ and so is a curvature.

Finally,equate the potential

$$V(x) = 2\sum_y g(x,y)$$

with k(x).

QED

Why interesting?

Linear algebra

A) We can easily generate large non-trivial integer matrices which are unimodular.

Hearing Euler

B) The number of positive eigenvalues minus the number of negative eigenvalues is the Euler characteristic.

Stable vaccum

C) We have a potential theoretic set-up, where all Green functions are bounded and have limited range.

Rich extensions

D) The situation links many topics and generalizes vastly, for example to k-point Green function set-ups.

Unit 7: Brouwer Lefschetz

The sum of the indices of a simplicial map T on a simplicial complex G is equal to the Lefschetz number $\chi_T(G)$, the super trace of the Koopman operator of T restricted to the harmonic forms of G.

7.42. A map $T : G \to G$ on a finite abstract simplicial complex G is a **simplicial map** if it maps the set V of 0-dimensional simplices into itself and if it is order-preserving, meaning that $x \subset y$ implies $T(x) \subset T(y)$. The map T does not have to be invertible. A constant map T that maps G to $x \in G$ for example is a simplicial map only if x is 0-dimensional.

7.43. If T is a simplicial map, its **attractor** $\bigcap_{k \geq 0} T^k(G)$ is a simplicial complex on which T is invertible. As we are interested in fixed points of T and fixed points must be in the attractor, we can assume T is invertible.

7.44. An invertible simplicial map T induces a **Koopman operator** U on the Hilbert space Λ of differential forms. It is $Uf(x) = f(Tx)\text{sign}(T|x)$. If T is invertible, it is a unitary operator. The linear map U commutes with the exterior derivative d and so commutes with the Hodge Laplacian. It therefore preserves the linear space of k-forms Λ_k and also preserves the harmonic k-forms.

7.45. Let U_k be the restriction of U to k-forms. Its super trace $\text{str}(U|\ker(H)) = \sum_{k=0}^{q} (-1)^k \text{tr}(U_k|\ker(H_k))$ of U is the **Lefschetz number** of T. We call it $\chi_T(G)$ because if T is the identity, then $\chi_T(G)$ is the Euler characteristic $\chi(G)$.

7.46. Let $F = \{x \in G, T(x) = x\}$ denote the **fixed point set**. T induces a permutation of the vertices of each fixed point x for which $\text{sign}(T|x)$ is the sign of. The number $i_T(x) = \omega(x)\text{sign}(T|x)$ is called the **index** of x.

7.47. The Lefschetz fixed point theorem is

Theorem: $\chi_T(G) = \sum_{x \in F} i_T(x)$.

Proof. The operator U commutes with the heat flow e^{-tL}. Define the operator $U_t = Ue^{-tL}$. The McKean-Singer symmetry implies that $\text{str}(U_t)$ is independent of t. For $t = 0$, it is the super trace of U which is $\sum_{x \in F} i_T(x)$ by the theorem below. Since $\lim_{t \to \infty} e^{-tL} = P$ is the projection on the harmonic forms, we have $\text{str}(Ue^{-tL}) \to \chi_T(G)$, proving the theorem. $\square$

Theorem: $\text{str}(U) = \sum_{x \in F} i_T(x)$

Proof. $\sum_{x \in G} \omega(x)U(x,x) = \sum_{x \in F} \omega(x)U(x,x) = \sum_{x \in F} \omega(x)\text{sign}(T|x) = \sum_{x \in F} i_T(x)$. $\square$

7.48. Lets see what happens if G is the complex of a complete graph K_n and T is a permutation of $V = \{1, \dots, n\}$. In this case $\mathrm{str}(U) = 1$ which is $\chi_T(G)$. The map U encodes the fixed points in the diagonal If the permutation belonging to T is $\pi = (c_1)(c_2) \dots (c_m)$, where c_i are the cycles, then every non-empty collection of cycles produces a fixed simplex of T.

7.49. The Lefschetz theorem for contractible spaces is **Brouwer's fixed point theorem**:

Theorem: A map on a contractible G has at least 1 fixed point.

Proof. If G is contractible, the cohomology only consists of constant 0-forms. The Lefschetz number is 1. □

7.50. Let $\mathrm{Aut}(G)$ denote the **automorphism group** of G, the set of symmetries. Given a subgroup A of $\mathrm{Aut}(G)$, the **average Lefschetz number** is $\chi(G/A) = \frac{1}{|A|} \sum_{T \in A} \chi_T(G)$. If G is trivial it is $\chi(G)$.

Theorem: If $H = G/A$ is a complex, $\frac{1}{|A|} \sum_{T \in A} \chi_T(G) = \chi(G/A)$.

Proof. Burnside's theorem for A acting on G gives $f_k(G/A) = \frac{1}{|A|} \sum_{T \in A} |F_k(T)|$ with $F_k(T) = \{x \in G_k, T(x) = x\}$ so that $\chi(G/A) = \sum_{k=0}^q (-1)^k f_k(G/A)$ is

$$\frac{1}{|A|} \sum_{T \in A} (-1)^k |F_k(T)| = \frac{1}{|A|} \sum_{T \in A} \sum_{x \in F} i_T(x) = \frac{1}{|A|} \sum_{T \in A} \chi_T(G) .$$

□

7.51. The **curvatures** of G/A are $\kappa(x) = \frac{1}{|A|} \sum_{T \in A} i_T(x)$. They add up to $\chi(G/A)$. We can see G as **branched cover** over $H = G/A$. If A is the cyclic group generated by T, then G is a $k : 1$ cover over $H = G/A$. The fixed points of T are branch points. If $T, T^2, \dots, T^{k-1}$ have no fixed points then $\chi_{T^l}(G) = 0$ and $\chi_{Id}(G) = \chi(G)$ so that $\chi(H) = \chi(G)/|A|$, a special **Riemann-Hurwitz** result.

7.52. Examples: 1) For $G = [0, 1, 2, 3]$ and $T(x) = T(3 - x)$, there is single fixed point $(1, 2)$ of index 1. 2) If G a figure 8 complex that carries an involution with 3 fixed points of index 1. The trace on harmonic 0 forms is 1. The trace on harmonic 1 forms is -2. 3) Invertible orientation preserving maps on a $2k$-sphere, or invertible orientation reversing maps on a $(2k + 1)$-sphere have at least 2 fixed points.

Brower-Lefschetz

G: finite abstract simplicial complex.

$T: G \to G$ is a simplicial map if $x \subset y \Rightarrow T(x) \subset T(y)$ and $T(V) \subset V$

$\chi_T(G) = \mathrm{str}(U_T | \ker(H))$ Lefschetz number

$Uf(x) = f(T(x))\mathrm{sign}(T|x)$ Koopman

$F = \{x \in G, T(x) = x\}$ fixed points

$i_T(x) = \mathrm{sign}(T|x)\omega(x)$ index

Lefschetz fixed point Theorem

$$\chi_T(G) = \sum_{x \in F} i_T(x)$$

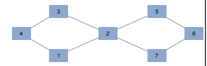
Brouwer fixed point Theorem

T on contractible G has fixed point

Example Figure 8 complex

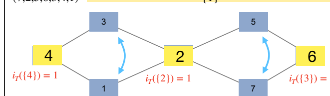
$G = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{1,2\},\{1,4\},\{2,3\},\{2,5\},\{2,7\},\{3,4\},\{5,6\},\{6,7\}\}$

$\mathrm{Aut}(G) = D_4$
dihedral group



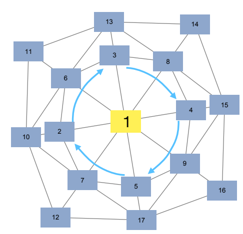
cycles		permutation T	$\chi_T(G)$
()	=	(1 2 3 4 5 6 7)	-1
(57)	=	(1 2 3 4 7 6 5)	1
(13)	=	(3 2 1 4 5 6 7)	1
(13)(57)	=	(3 2 1 4 7 6 5)	3
(15)(37)(46)	=	(5 2 7 6 1 4 3)	1
(1537)(46)	=	(5 2 7 6 3 4 1)	1
(1735)(46)	=	(7 2 5 6 1 4 3)	1
(17)(35)(46)	=	(7 2 5 6 3 4 1)	1

T	indices $i_T(x)$
(1,2,3,4,5,6,7)	$\{1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1\}$
(1,2,3,4,7,6,5)	$\{1,1,1,1,1,-1,-1,-1,-1\}$
(3,2,1,4,5,6,7)	$\{1,1,1,1,1,-1,-1,-1,-1\}$
(3,2,1,4,7,6,5)	$\{1,1,1\}$
(5,2,7,6,1,4,3)	$\{1\}$
(5,2,7,6,3,4,1)	$\{1\}$
(7,2,5,6,1,4,3)	$\{1\}$
(7,2,5,6,3,4,1)	$\{1\}$

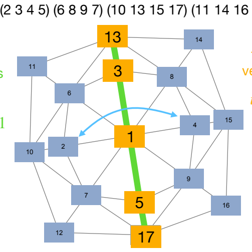


Example Ball

$\mathrm{Aut}(G) = D_4$ $\chi_T(G)$=1 for all T



T= (2 3 4 5) (6 8 9 7) (10 13 15 17) (11 14 16 12)



T=(2 4) (6 8)(7 9)10 15)(11 14)(12 16)

Proof T is invertible on

K= $\bigcap_{k \geq 0} T^k(G)$ the attractor

$Uf(x) = f(T(x))\mathrm{sign}(T|x)$
unitary Koopman operator

Heat deformation argument:

$\mathrm{str}(Ue^{-tH})$ is t independent
(McKean-Singer symmetry)

t=0 $\mathrm{str}(U) = \sum_{x \in F} i_T(x)$

add up $\omega(x)\mathrm{sign}(T|x)$ over diagonal of U, if x is a fixed point.

$t \to \infty$ $\mathrm{str}(Ue^{-tH}) \to \chi_T(G)$

H and U commute. QED

Quotient

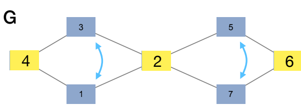
G: finite abstract simplicial complex.
$A \subset \mathrm{Aut}(G)$ subgroup

$\frac{1}{|A|} \sum_{T \in A} \chi_T(G)$ average Lefschetz number

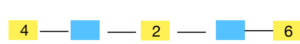
$$\chi(G/A) = \frac{1}{|A|} \sum_{T \in A} \chi_T(G)$$

Example: A={1,T} with T=(13)(57)
$\chi_T(G) = 3$ $\chi_1(G) = -1$
average (3-1)/2 = 1

G



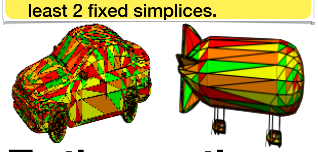
G/A path complex $\chi(G/A) = 1$



G is a branched cover over G/A!

On spheres:

An orientation preserving map of an object with the topology of a 2-sphere has at least 2 fixed simplices.



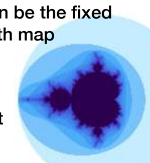
To the continuum

The discrete implies the continuum

but more is needed:
G has manifold structure
T smooth
isolated fixed points

Every closed set can be the fixed point set of a smooth map

The Mandelbrot set for example is the fixed point set of a smooth map



Unit 8: Sphere formula

If G is any finite abstract simplicial complex, then $\sum_{x \in G} \omega(x)\chi(S(x)) = 0$. If all unit spheres have the same non-zero Euler characteristic, like for odd-dimensional manifolds, then the Euler characteristic of the complex is zero.

8.53. Let G be a finite abstract simplicial complex. The **unit sphere** $S(x)$ of $x \in G$ is the boundary of the **star** $U(x) = \{y \in G, x \subset y\}$, the smallest open set that contains x. As this is $\overline{U}(x) \setminus U(x)$, the unit sphere is a simplicial complex. The topological definition links it to a natural topology $\mathcal{O}$ on G, even so it is never Hausdorff, if the dimension of G is positive.

8.54. For a vertex v in a graph, the unit sphere $S(v)$ is the graph generated by the neighbors of v. For a simplex x in a complex G, the unit sphere is $S(x) = \delta U(x)$. There is a connection: if $\Gamma(G) = (V, E)$ is the graph with vertices $V = G$ and $E = \{(x, y), x \subset y \text{ or } y \subset x\}$, one can associate $S(x)$ with $\Gamma(S^+(x)) \oplus \Gamma(S^-(x))$. We write $S(x) = S^+(x) \oplus S^-(x)$ even-so we join an open with a closed set. This is a form of hyperbolicity because for a hyperbolic fixed point of a smooth map T in $\mathbb{R}^n$, we have **stable and unstable manifolds** $W^-(x)$ and $W^+(x)$ which when intersected with a small sphere $S(x) = S_r(x)$ produce two spheres $S^-(x)$ and $S^+(x)$ which have the property that their join is $S(x)$. In G, the hyperbolic structure comes from the **dimension functional** dim on G. Here is the **hyperbolicity lemma**:

Theorem: $S(x) = S^+(x) \oplus S^-(x)$ for all $x \in G$.

Proof. We show the graph join $\Gamma(S(x)) = \Gamma(S^+(x)) \oplus \Gamma(S^-(x))$. A simplex $y \in G$ is a vertex in $\Gamma(S(x))$ if either y is a subset of some $z \in S(x))$ meaning y is in $S^-(x)$ or if y contains some $z \in S(x)$ meaning y is in $S^+(x)$. So, the vertex sets of the two graphs agree. If two points y, z in $\Gamma(S(x))$ are connected, we either have $y \subset z$ or $z \subset y$. There are four cases: either both are in $S^-(x)$ which means they are connected in $S^-(x)$ and so connected in the join. The same hold if both are in $S^+(x)$. Any pair $y \in S^-(x)$ and $z \in S^+(x)$ satisfies $y \subset x \subset z$ so that they are connected in $S(x)$. $\square$

Theorem: $\sum_{x \in G} \omega(x)\chi(S(x)) = 0$.

Proof. (i) For any x we have the local valuation formula $\chi(B(x)) = \chi(U(x)) + \chi(S(x))$. Proof: For any subsets $A, B \subset G$, whether open or closed or neither, we have the valuation formula $\chi(A) + \chi(B) = \chi(A \cup B) - \chi(A \cap B)$ because each of the basic valuations $f_k(G)$ counting the number of k-dimensional simplices satisfies the formula and χ is a linear combination of such basic valuations. (ii) $\chi(B(x)) = 1$ for all x. Proof: Use induction with respect to the number of elements in $B(x)$. If $B(x)$ has one element, it has $\chi(B(x)) = 1$. We can reduce the size of $B(x)$ by taking a way an element $y \in B(x)$ different from x. The complex $B'(x) = B(x) \setminus U(y)$ is now a unit

ball $B'(x)$ in a smaller $G \setminus U(y)$. The induction assumption assures that $\chi(B'(x)) = 1$. The local valuation formula gives that $\chi(B(x))) = \chi(B'(x)$.
(iii) the energy formula $\sum_{x \in G} \omega(x)\chi(U(x)) = \chi(G)$ was proven in the unimodularity unit. It told that the super trace of g was Euler characteristic. Here is an other proof: Replace the ± 1-valued function $\omega(x)$ by a more general function $h(x)$ and define $\chi_h(A) = \sum_{x \in A} h(x)$ and $\psi_h(x) = \omega(x)\chi_h(U(x))$. Both maps $h \to \chi_h(A)$ and $h \to \psi_h(A)$ are linear in h for fixed A so that we only need to consider the case where $h(x_0) = 1$ and $h(x) = 0$ for $x \neq x_0$. The right hand side is then $\chi_h(G) = 1$. The left hand side is $\sum_{x \in G} \omega(x)\chi_h(U(x)) = \sum_{x \in G, x \subset x_0} \omega(x)$, which is the Euler characteristic of the simplicial complex $\overline{\{x_0\}} = \{x \subset x_0\}$ which is always 1. (iv) $\omega(G) = \sum_{x \in G} \omega(x)\chi(B(x)) = \sum_{x \in G} \omega(x) = \chi(G)$. $\square$

8.55. The value $g(x,x) = \omega(x)^2\chi(U(x) \cap U(x) = \chi(U(x))$ is a diagonal entry in the **Green function**. It is the **self-energy** of $x \in G$. Now $\chi(S(x)) = 1 - \chi(U(x)$ gives $\sum_{x \in G} \omega(x)\chi(S(x)) = \sum_{x \in G} \omega(x)(1 - \chi(U(x)) = \chi(G) - \chi(G) = 0$, which is the sphere formula.

8.56. What is $\sum_{x \in G} \chi(S(x))$? This is the trace of $L - g$. We called this the **Hydrogen functional** because of formal similarity with the quantum mechanical Hamiltonian of the Hydrogen atom, where the potential part involves the kernel of the inverse of the kinetic part - the $1/r$ potential being determined by the Laplacian Δ in $\mathbb{R}^3$ for example. We called $L - L^{-1} = L - g$ the **Hydrogen operator**.

Theorem: $\sum_{x \in G} \chi(S(x)) = \mathrm{tr}(L - L^{-1})$.

8.57. The Euler-Gem and sphere formula together immediately give the zero Euler characteristic result:

Theorem: All odd-dimensional manifolds satisfy $\chi(G) = 0$.

Proof. Euler-Gem formula assures that odd-dimensional manifolds have unit spheres with $\chi(S(x)) = 2$. The sphere formula implies $0 = \sum_x \omega(x)\chi(S(x)) = \sum_x \omega(x)2 = 2\chi(G)$. $\square$

Sphere Formula

G: finite abstract simplicial complex.

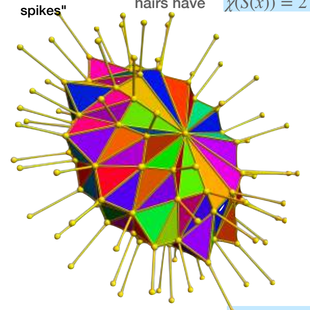
$U(x) = \{y \in G, x \subset y\}$ star

$S(x) = \delta U(x) = \overline{U(x)} \setminus U(x)$ unit sphere

$$\sum_{x \in G} \omega(x)\chi(S(x)) = 0$$

Example: "manifold with spikes"

tips have $\chi(S(x)) = 1$
follicles have $\chi(S(x)) = 1$
hairs have $\chi(S(x)) = 2$



edges within the manifold have $\chi(S(x)) = 0$
triangles within the manifold have $\chi(S(x)) = 0$

Hyperbolicity

G: finite abstract simplicial complex.

$S^+(x) = \{y \in G, x \subset y, x \neq y\}$

$S^-(x) = \{y \in G, y \subset x, x \neq y\}$

stable and unstable spheres.

S^+ open and S^- is closed.

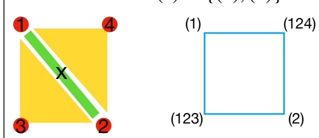
the join of the two complexes is only a delta set but their graph join is S(x).

$S(x) = S^+(x) \oplus S^-(x)$ join

x=(12) U(x)={(12),(123),(124)}

$S^+(x) = \{(123), (124)\}$

$S^-(x) = \{(1), (2)\}$



$S(x) = \{(1), (2), (123), (124)\}$

Its graph is C_4

Picture

$S^+(x) = \{y \in G, x \subset y, x \neq y\}$

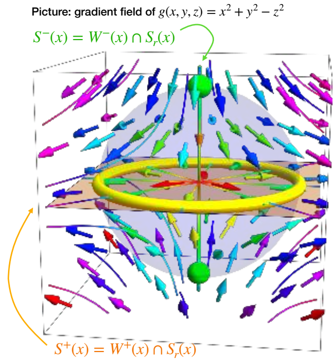
$S^-(x) = \{y \in G, y \subset x, x \neq y\}$

$S(x) = S^-(x) \oplus S^+(x)$

stable and unstable manifolds intersected with small sphere.

Picture: gradient field of $g(x, y, z) = x^2 + y^2 - z^2$

$S^-(x) = W^-(x) \cap S_r(x)$



$S^+(x) = W^+(x) \cap S_r(x)$

In the simplicial complex case g(x) = dim(x)

Proof:

G: finite abstract simplicial complex.

$B(x) = \overline{U(x)}$ unit ball

$S(x) = B(x) \setminus U(x)$ unit sphere

$B(x) = U(x) \cup S(x)$

disjoint union of an open and closed set. By counting:

$\chi(B(x)) = \chi(U(x)) + \chi(S(x))$

Therefore $\chi(U(x)) = g(x, x)$ the self-energy of a simplex x is

$\chi(U(x)) = 1 - \chi(S((x))$

We have seen that $\omega(x)(1 - \chi(S(x))$ was a curvature adding up to $\chi(G)$

But, $\sum_x \kappa(x) = \chi(G)$ is equivalent to the sphere formula.

Manifolds

Odd dimensional compact closed manifolds G have $\chi(G) = 0$.

Henri Poincaré 1814-1895

Proof: all S(x) are even dimensional spheres. Euler's Gem shows: $\chi(S(x))$=2

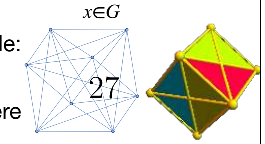
The sphere formula

$$\sum_{x \in G} \omega(x)\chi(S(x)) = 0$$

gives $$\sum_{x \in G} \omega(x) = 0$$

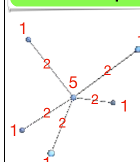
Example:

$K_{2,2,2,2}$ 3-sphere



Example: Star graph

$G = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}\}$

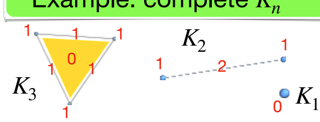


$\chi(S(x)) \in \{-2, 1, 5\}$

Total Euler characteristic of points: 10

Totel Euler characteristic of edges: 10

Example: complete K_n



Example: Poincare Sphere

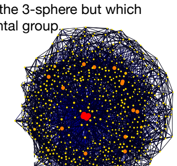
The Poincaré sphere is a 3-manifold that has the same cohomology than the 3-sphere but which has a different fundamental group.

$\chi(S(x)) = 2, \forall x \in G$

Implementation with f=(24, 154, 260, 130)

b=(1,0,0,1)

Betti vector



Unit 9: Level sets

If G is a discrete q-manifold and $g : V \to K_k = \{0, \dots, k\}$ is a function, then $M_g = \{x \in G | g(x) = K_k\}$ is a $(q-k)$-manifold if not empty.

9.58. A q**-manifold** is a finite abstract simplicial complex G such that every unit sphere $S(x)$ is a $(q-1)$-sphere. A q**-sphere** is a q-manifold G such that $G \setminus U(x)$ and $S(x)$ are both contractible. A complex G is **contractible** if there exists x such that both $G \setminus U(x)$ and $S(x)$ are both contractible. The initial object **void** 0 is the (-1) sphere and the terminal object 1 is contractible.

9.59. The vertex set $V = \bigcup_{x \in G} x$ in G can be identified with the 0-dimensional objects G_0 in G. A function $g : V \to K_k = \{0, \dots, k\}$ defines a **level set** of G denoted by $M_g = \{x \in G, g(x) = K_k\}$, where $g((x_0, x_1, \dots, x_)) = \{g(x_0), \dots g(x_k)\}$ In the case $k = 1$ for example, this is the set of x on which $g(x) = \{0, 1\}$ takes both values, meaning that $\{v \in x, g(v) = 0\}$ and $\{v \in x, g(v) = 1\}$ are both non-empty. The set M_g is a priori only open. This means $x \in M_g$, and $y \in G$ contains x then $y \in M_g$. But M_g defines a simplicial complex: take the set M_g as the vertices of a graph Γ and connect two vertices (a, b) if $a \subset b$ or $b \subset a$. The Whitney complex of this graph is then a simplicial complex and if we say M_g is a manifold, we mean that its simplicial complex is.

9.60.

Theorem: Any $g : V(G) \to K_k$ on a q-manifold G has a level set M_g that is a $q - k$ manifold if it is not empty.

9.61. The proof uses the hyperbolic structure of any simplicial complex. In the case of manifolds, we can even deal with a Morse complex because every $x \in G$ is a critical point of $x \to \dim(x)$. The Morse index of x is $m(x) = \dim(S^-(x)) = k$, where $S^-(x) = \{y \in G, y \subset x, y \neq x\}$. Complementary is the $S^+(x) = \{y \in G, x \subset y, y \neq x\}$. G is a manifold if and only if every $S^+(x)$ is a sphere. Again, $S + (x)$ is only an open set at first and we refer to the Whitney complex of its graph. Hyperbolicity is:

Theorem: If G is a q-manifold and $\dim(x) = k$, then $S(x)$ is the join of the $(k-1)$-sphere $S^-(x)$ and $(q-k-1)$ sphere $S^+(x)$.

Proof. The graph of $S(x)$ contains both simplices $S^-(x)$ strictly contained in x and simplices $S^+(x)$ that strictly contain x. Every $y \in S^-(x)$ is contained in any $z \in S^+(x)$. □

9.62. In the special case, when G is the boundary complex $S^-(x)$ of a k-simplex x, level sets are always spheres. For the 2-simplex $x = K_3$ for example, the boundary sphere is $S^-(x) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$. For any function $g : \{1, 2, 3\} \to \{0, 1\}$ the level set is always a 0-sphere.

9.63. A special case is if G be a complete n-dimensional complex. Every surjective $g : V(G) \to K_k$ produces a level set M_g that is a $(n-k)$-sphere. The sets $V_j = \{g = j\} \subset V$ partition V into $k+1$ sets of size $n_j > 0$ such that the sum is $n+1$. The level set M_g is the join of all these $(n_j - 1)$-spheres which is a $(n-k)$ - sphere because $(\sum_j n_j) - 1 = n + 1 - k - 1 = n - k$.

9.64. Now the proof of the main theorem:

Proof. We use induction with respect to dimension q. The unit sphere $S_G(x)$ in G is the join of $S^+(x)$ and $S^-(x)$. The unit sphere $S_M(x)$ in G is the join of $S^+_M(x)$ and $S^-_M(x)$. We have $S^+_M(x) = S^+(x)$ because if y is in $S(x)$ and g takes all values on x, then g takes all values on y so that y is also in $S^+_M(x)$. As seen above, the level surface in the $k-1$ sphere $S^-(x)$ is $k-2$-sphere. So $S_M(x)$ is the join of a $(k-2)$ sphere with a $q-k-1$ sphere which is a $q-k-1$ sphere □

9.65. The set-up can be generalized if K_k is replaced with a $(k+1)$-partition complex $P = K_{n_0,n_1,...n_k}$. If a function $g : V(G) \to V(P)$ is given, define $M_g = \{x \in G, g(x)$ contains at least one facet of $P\}$. In that case again M_g is a $q-k$ manifold if it is not empty. Every facet f of P now produces a patch $M_{g,f} = \{x \in g, f \subset g(x)\}$ which is a $q-k$ manifold with boundary. The union of all these patches is the manifold M_g without boundary.

Theorem: Any $g : V(G) \to P$ from a q-manifold G to a **partition complex** P of dimension k has a level set M_g that is a $q-k$ manifold or $\emptyset$.

9.66. In the continuum, level sets in q-manifolds are not manifolds in general, like for $g(x,y) = \sin(x)\sin(y) = 0$ on the 2-manifold $M = \mathbb{T}^2$. The Morse Sard theorem assures that for almost all c in the range of $g : M \to \mathbb{R}^k$, the level set $\{g = c\}$ is a $q-k$ manifold. In the discrete, no regularity at all is needed.

9.67. Examples:

- If g is 1 on some vertex v and 0 else, then $M_g = S(v)$ is the unit sphere.
- If G is a hexagonal lattice and $g : V \to \{0,1\}$ is a function, then M_g consists of all triangles and edges on which g takes two values. If an edge e is in M_g then the two attached triangles are there. If t is a triangle in M_g, then there are exactly two edges contained in M_g where g takes two values. Since every point in M_g has exactly two neighbors, the level set must be a union of circular graphs.

Level Sets

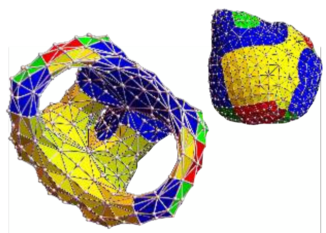
G: finite abstract simplicial complex with vertex set V and dimension q.

$g : V \to K_k = \{0, \dots, k\}$ function

$M_g = \{x \in G, g(x) = K_k\}$

M_g is open but defines a graph that has a simplicial complex as Whitney complex.

For any q-manifold G and any function $g : V \to K_k$, the level set M_g is a (q-k)-manifold or empty.



Proof

Induction in q. Use hyperbolicity:

$S(x) = S^+(x) \oplus S^-(x)$ for a n-simplex x

$S^-(x)$ is a (n-1)-sphere, the boundary complex of x.

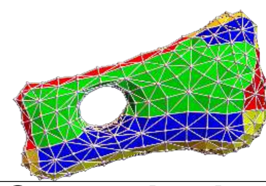
A $S_g^-(x)$ is a (n-k-1)-sphere

B $S_g^+(x) = S^+(x)$ (q-n-1)-sphere

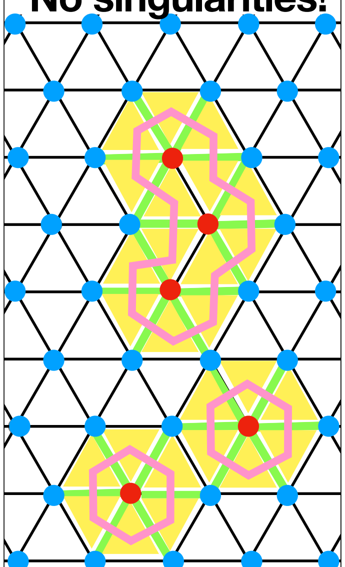
Proof: $S_g^+(x) \subset S^+(x)$

$y \in S^+(x)$ implies $y \in S_g^+(x)$

C $S_g(x) = S_g^+(x) \oplus S_g^-(x)$ is a (q-k-1) -sphere



No singularities!

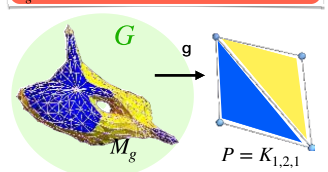


Generalization

G: q-manifold with vertex set V.

$g : V \to P = K_{n_0,\dots,n_k}$ partition complex

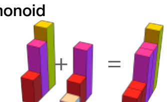
$M_g = \{x \in G, g(x) \text{ contains facet in P}\}$



If not empty, M_g is a (q-k)-manifold. It is union of patches, belonging to facets reached by g.

Partition complexes form a commutative join monoid

Example:

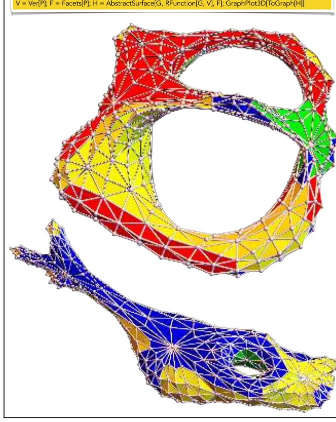


$K_{2,4,5} \oplus K_{1,2,4} = K_{1,2,2,4,4,5}$

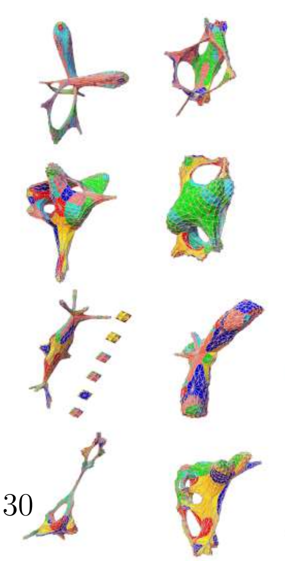
Code

The implementation is straightforward:

```
Generate[A_] :=  If[A == {}, {}, Sort[Delete[Union[Sort[Flatten[Map[Subsets, A], 1]]], 1]]];
Whitney[s_] := Generate[FindClique[s, Infinity, All]]; L = Length;  Ver[X_] :=Union[Flatten[X]];
RFunction[G_, P_] := Module[{}, R[x_] := x -> RandomChoice[Range[L[Ver[P]]]]; Map[R, Ver[G]]];
Facets[G_] :=Select[G, (L[#] ==Max[Map[L, G]]) &];  Poly[X_]:=PolyhedronData[X, "Skeleton"];
AbstractSurface[G_, f_, A_] := Select[G, (Sum[If[SubsetQ[# /. f, A[[l]]], 1, 0], {l, L[A]}] > 0) &];
Z[n_] := Partition[Range[n], 1]; ZeroJoin[a_] := If[L[a] == 1, Z[a[[1]]], Whitney[CompleteGraph[a]]];
CJoin[G_,H_]:=Union[G,H+Max[G]+1,Map[Flatten,Map[Union,Flatten[Tuples[{G,H+Max[G]+1}],0]]]];
ToGraph[G_]:=UndirectedGraph[n=L[G]; Graph[Range[n], Select[Flatten[Table[k ->l, {k, n}, {l, k+1, n}],
1], (SubsetQ[G[[#[[2]]]], G[[#[[1]]]]]) &]]];
J = Whitney[Poly["Icosahedron"]]; G=CJoin[J, Whitney[CycleGraph[13]]]; P=ZeroJoin[{2,2,3}];
V = Ver[P]; F = Facets[P]; H = AbstractSurface[G, RFunction[G, V], F]; GraphPlot3D[ToGraph[H]]
```



Random manifolds



Unit 10: Index formula

If g is a vertex coloring of a graph (V, E), the level set $S_g(v) \subset S(v)$ produces the **symmetric Poincare-Hopf index** $j_g(v) = 1-\chi(S(v))/2-\chi(S_g(v))/2$. This implies that odd-dimensional manifolds are flat and that for 4-manifolds, curvature $K(v)$ is the genus expectation of a random 2-manifold in the 3-manifold $S(v)$.

10.68. If a finite abstract simplicial complex G defines a graph (V, E) the Poincaré-Hopf theorem for G is just the Poincaré-Hopf theorem for the corresponding graph with $V = G$. A locally injective function $g : G \to R$ is a vertex coloring function $g : V \to R$. The index $i_g(v) = 1 - \chi(S_g^-(v))$ defines then the **symmetric index** $j_g(v) = [i_g(v)+i_{-g}(v)]/2$. Since $\text{sign}(g(w)-g(v))$ is a $\{-1, 1\}$-valued function on $S(v)$, we can look at its level set $S_g(v)$ in $S(v)$. If $S(v)$ was a $(q-1)$-sphere, then $S_g(v)$ is a $(q-2)$-manifold.

10.69. The following result combines Poincaré-Hopf with the discrete Sard story. It shows that for manifolds, the symmetric index is expressible as the Euler characteristic of sub-manifolds. First for general complexes:

Theorem: If g is a coloring, then $j_g(v) = 1 - \chi(S(v))/2 - \chi(S_g(v))/2$.

Proof. The level set result clarifies what we mean with $S_g(v)$: it is a level set in $S(v)$. We have a decomposition $S(v) = S_g^-(v) \cup S_g^+(v) \cup S_g(v)$ in the sense that every simplex in $S(v)$ is either a simplex in $S_g^-(v)$ or a simplex in $S_g^+(v)$ or a simplex in $S_g(v)$. Therefore, $\chi(S(v)) = \chi(S_g^-(v)) + \chi(S_g^+(v)) - \chi(S_g(v))$, the negative sign coming from the fact that $S_g(v)$ has edges as smallest dimensional simplices, shifting dimensions by 1. Therefore, $\chi(S(v)) + \chi(S_g(v)) = 1 - i_g(v) + 1 - i_{-g}(v) = 2 - 2j_g(v)$. Now solve for $j_g(v)$. ☐

10.70. In the case of manifolds, we have by the Euler gem formula for odd q that $j_g(v) = -\chi(S_g(v))/2$ which means that $j_g(v) = 0$. Also the sphere formula had shown that the odd-dimensional manifold $S_g(v)$ had zero Euler characteristic. Therefore, for any index expectation curvature that comes from a probability measure, for which $g \to -g$ is an isometry we have:

Theorem: Odd-dimensional manifolds are flat.

10.71. In the case of even-dimensional manifolds we have $j_g(v) = 1 - \chi(S_g(v)))/2$. Motivated from notions in two dimensions, let us call $g = 1 - \chi(G)$ the **genus** of G. It is also known as a **reduced Euler characteristic**.

Theorem: Curvature $K(v)$ in even dimensions is the genus average of $q-2$ manifolds in $S(v)$.

10.72. For a 4-manifold for example, where $S_g(v)$ is an orientable manifold of genus g with Euler characteristic $\chi = 2 - 2g$, we have:

Theorem: For a 4-manifold, the curvature at v is the expected genus of a random surface in $S(v)$.

10.73. The theorem generalizes from manifolds **Dehn-Sommerville q-manifolds**. These are q-dimensional finite abstract complexes for which all $S(x)$ are Dehn-Sommerville $(q-1)$-manifolds of Euler characteristic $1-(-1)^q$. **Euler's gem** implies that q-manifolds are q-Dehn-Sommerville manifolds. A **Dehn-Sommerville q-sphere** is a Dehn-Sommerville q-manifold of Euler characteristic $1+(-1)^q$. The sphere formula $\sum_{x \in G} \omega(x)\chi(S(x)) = 0$ implies that every odd-dimensional Dehn-Sommerville manifold is a Dehn-Sommerville sphere.

Theorem: A level subset defined by a function $g : V(G) \to \{0, \dots, k\}$ on a Dehn-Sommerville q-manifold G is a $(q-k)$ Dehn-Sommerville manifold or empty.

10.74. It implies :

Theorem: For odd-dimensional Dehn-Sommerville complexes, $j_g(v) = 0$ for all v. For even dimensional Dehn-Sommerville complexex $j_g(v) = (1 - \chi(S_g(x))/2$.

10.75. Let $(\Omega, \mathcal{A}, P)$ be a probability measure on locally injective functions with the property that $Tg = -g$ is an automorphism of the probability space. We call the index expectation curvature $k(v) = \mathrm{E}[j_g(v)]$ a **symmetric curvature**. The Levitt curvature and the color curvature are both symmetric. We have in particular:

Theorem: Odd-dimensional Dehn-Sommerville manifolds are flat.

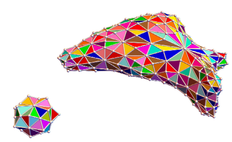
Index Formula

g: locally injective function on (V,E)
$S_g(v) = \{g - g(v) = 0\} \subset S(v)$

If $i_g(v)$ is the Poincare-Hopf index,
$j_g(v) = [i_g(v) + i_{-g}(v)]/2$
is called symmetric index.

Theorem: (2012)

$$j_g(v) = 1 - \frac{\chi(S(v))}{2} - \frac{\chi(S_g(v))}{2}$$

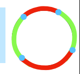


We see random $S_g(v)$ in a 4-manifold.

Proof:

start with decomposition :

$$S(v) = S_g^+(v) \cup S_g^-(v) \cup S_g(v)$$



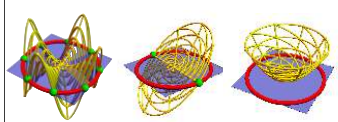
start with decomposition : $S_g(v)$ has dimension shifted, because edges are smallest units.
Euler characteristic:

$$\chi(S(v)) = \chi(S_g^+(v)) + \chi(S_g^-(v)) - \chi(S_g(v))$$

$$1 - \frac{\chi(S(v))}{2} = \frac{1}{2} - \frac{\chi(S_g^+(v))}{2} + \frac{1}{2} - \frac{\chi(S_g^-(v))}{2} + \frac{\chi(S_g(v))}{2}$$

$$1 - \frac{\chi(S(v))}{2} = j_g(v) + \frac{\chi(S_g(v))}{2}$$

Solve $j_g(v) = 1 - \frac{\chi(S(v))}{2} - \frac{\chi(S_g(v))}{2}$ QED



$j_g(v) =$ 1-0/2-8/2 =-3 $j_g(v) =$ 1-0/2-2/2 =0 $j_g(v) =$ 1-0/2-0/2 =1

Continuum

If $g : M \to \mathbb{R}$ is smooth function and v is a critical point, for a small enough sphere S(v)

$$j_g(v) = 1 - \frac{\chi(S(v))}{2} - \frac{\chi(S_g(v))}{2}$$

For odd dimensional manifolds,

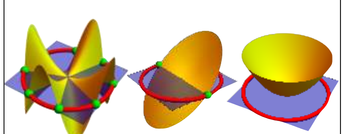
$$j_g(v) = 1 - \frac{\chi(S(v))}{2} - \frac{\chi(S_g(v))}{2} = 0$$

For even dimensional manifolds,

$$j_g(v) = 1 - \frac{\chi(S(v))}{2} - \frac{\chi(S_g(v))}{2}$$

$$= 1 - \frac{\chi(S_g(v))}{2}$$

Euler characteristic of a 4 manifold is the difference of the volume and a Hilbert-Einstein type action.



Odd dimensional manifolds are flat

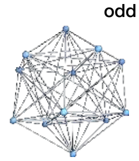
k(v) index expectation curvature that is $g \leftrightarrow -g$ invariant

Odd dimensional manifolds with such curvature are flat.

Proof: $S_g(v)$ is odd-dim manifold

$$j_g(v) = 1 - \frac{\chi(S(v))}{2} - \frac{\chi(S_g(v))}{2}$$

are both zero for odd dim manifolds QED



For any 5-manifold:
K=-E/6 + F/4 - C/6
is constant 0.

$f_G(t) = 1 + 12t + 60t^2 + 160t^3 + 240t^4 + 192t^5 + 64t^6$

Curvature as an average genus

Define $1 - \chi(G)/2$ the genus of G.

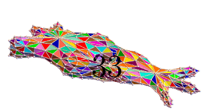
For even dimensional manifolds: curvature is the average genus of the level set $S_g(v)$ in S(v).

Proof: $S_g(v)$ is even-dim manifold

$$j_g(v) = 1 - \frac{\chi(S_g(v))}{2} - \frac{\chi(S(v))}{2}$$

genus of level set in S(v) zero for even dim manifolds

curvature is an expectation of $j_g(v)$.

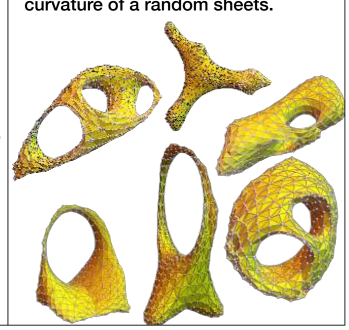


4 manifolds

$$\kappa(v) = E[1 - \frac{\chi(S_g(v))}{2}]$$

$$\chi(S_g(v)) = \sum_{w \in S_g(v)} K(w)$$

$\chi(G)$ -|V(G)| is expectation of curvatures of random 2D-surface curvatures. Compare: Hilbert-Einstein action is the expectation of sectional curvature of a random sheets.



Unit 11: Quadratic cohomology

If G is a finite abstract simplicial complex, the quadratic characteristic $\chi_2(G) = \sum_{x \cap y \in G} \omega(x)\omega(y)$ is a quadratic version of Euler characteristic $\chi(G) = \sum_x \omega(x)$. It comes with a cohomology similarly as $\chi(G)$ comes with simplicial cohomology. For q-manifolds with boundary $\chi_2(G) = \chi(G) - \chi(\delta G)$.

11.76. Classical theorems like Gauss-Bonnet, Poincare-Hopf, Euler-Poincare or Brouwer-Lefschetz are related to Euler characteristic $\chi(G) = \sum_{x \in G} \omega(x)$, both in the continuum as in the discrete. There are higher order theories for which **quadratic characteristic** $\chi_2(G) = \sum_{x \cap y \in G} \omega(x)\omega(y)$ is the simplest. Some theorems generalize directly but there are changes, the most important being that χ_2 is of a topological nature. It is not homotopy invariant. If G is the closure of $\{x\}$ then $\chi_2(G) = \omega(x) = (-1)^q$. For manifolds without boundary, $\chi_2(G) = \chi(G)$.

11.77. Quadratic cohomology works with functions on **pairs of intersecting simplices** $(x, y) \in G \times G$. The **exterior derivative** is defined as $df(x, y) = d_x f(x, y) + \omega(x) d_y(f, y)$, where d_x, d_y are the partial simplicial exterior derivatives. Cubic, quartic and higher characteristics and cohomology are defined similarly.

11.78. With $\dim(x, y) = \dim(x) + \dim(y)$, let $\Lambda_k = \{f : G_k \to \mathbb{R}$, where $G_k = \{(x, y), \dim(x, y) = k\}$ is the space of k-forms and let f_k denote its dimension. Then $w(G) = \sum_{k=0}^{2q} (-1)^k f_k$. The space $\Lambda = \oplus_{k=0}^{2q} \Lambda_k$ of **differential forms** has as dimension the number of ordered pairs $(x, y) \in G^2$ with $x \cap y \in G$. It is a finite dimensional vector space and $\mathrm{str}(1) = \chi_2(G)$ is the quadratic characteristic. The cardinalities of the quadratic complex is encoded by $f_{kl} = \{(x, y), x \cap y \neq \emptyset, \dim(x) = k, \dim(y) = l\}$.

11.79. The Dirac operator $D = d + d^*$ and the Hodge operator $H = D^2 = dd^* + d^*d$ are defined as before. If G has maximal dimension q, then H has $2q + 1$ blocks H_k. It belongs to forms $f(x, y)$ acting on pairs (x, y) of simplices with $\dim(x) + \dim(y) = k$. The kernel of H_k is called the k'th Betti number b_k. As in the linear case, we can split $\Lambda(G)$ into **even forms** Λ_{2j} and **odd forms** Λ_{2j+1}. The matrix D is an isomorphism between even and odd forms on the image of H so that $\mathrm{str}(H^n) = 0$ for $n > 0$. The analog of Euler-Poincaré is proven in exactly the same way using heat deformation.

Theorem: $\chi_2(G) = \sum_{k=0}^{2q} (-1)^k f_k = \sum_{k=0}^{2q} (-1)^k b_k$.

11.80. An automorphism T of G also induces an automorphism on the intersection diagonal of $G \times G$ by $T(x, y) = (T(x), T(y))$ the reason being that if $x \cap y \neq \emptyset$, then also $T(x) \cap T(y) = T(x \cap y) \neq \emptyset$. The Lefschetz number $\chi_{2,T}(G)$ is as in the linear case the super trace of the Koopman operator U on cohomology. The index of a fixed point (x, y) is $i_T(x, y) = \omega(x)\omega(y)\mathrm{sign}(T|x)\mathrm{sign}(T|y)$. Also the Lefschetz fixed point theorem generalizes with the same heat proof to a **quadratic Lefschetz fixed point theorem**:

Theorem: $\chi_{2,T}(G) = \sum_{(x,y) \in F} i_T(x,y)$.

11.81. It generalizes the Euler-Poincaré theorem because for $T = Id$, $\chi_{2,T}(G) = \chi_2(G)$ and every (x,y) with $x \cap y \neq \emptyset$ is then a fixed point of T with index $\omega(x)\omega(y)$ so that the right hand side in the Lefschetz theorem is the definition of $\chi_2(G)$.

11.82. Also Gauss-Bonnet, Poincaré-Hopf and index expectation generalize. Curvature $K(v)$ for quadratic valuation is $K(v) = \sum_{x \sim y} \omega(x)\omega(y)/|x|$.

Theorem: $\chi_2(G) = \sum_{v \in V} K(v)$.

11.83. For manifolds with boundary, we know how to compute it.

Theorem: $\chi_2(G) = \chi(G) - \chi(\delta G)$.

For even q, this gives $\chi_2(G) = \chi(G)$ even with boundary. For odd q, this gives $\chi_2(G) = -\chi(\delta G)/2$.

11.84. Examples: 1) For $P_2 = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\}\}$, there 15 interacting pairs (x,y) where $f_0 = 3, f_1 = 8, f_2 = 4$. The Wu characteristic is $3 - 8 + 4 = -1$. 2) The utility graph $G = K_{3,3}$ has Euler characteristic $\chi(G) = -3$ and Wu characteristic $\chi_2(G) = 15$. It is maximal among all graphs with 6 vertices. 3) For a star graph with n rays, the Wu characteristic is $n^2 - 3n + 1$. For example, for $n = 0$, it is 1, for $n = 1$ it is -1 for $n = 9$ it is 55. All star graphs are contractible illustrating the non-homotopy invariance. 4) Adding a hair to a 2 sphere reduces the Wu characteristic by 2. 5) The Wu characteristic of the cube graph is 20, the Wu characteristic of the dodecahedron is 50. Both graphs have no triangles and constant vertex degree $d = 3$ so that in both cases, the curvature is constant $5/2$. 6) The dunce hat has quadratic cohomology $b = (0,0,0,1,2)$. 7) The Moebius strip has $b = (0,0,0,0,0)$ while the cylinder has $b = (0,0,1,1,0)$. The cohomology sees the twist. The Lefschetz fixed point theorem implies that if an automorphism preserves harmonic 2-forms and reverses harmonic 3-forms, then there are at least 2 fixed points. We once speculated that the preservation of harmonic 2-forms could be related to area preservation and the twisting of harmonic 3 forms be related to twist so that quadratic cohomology could give Birkhoff's fixed point theorem.

Quadratic Cohomology

G finite abstract simplicial complex.

$$\chi_2(G) = \sum_{x \cap y \in G} \omega(x)\omega(y)$$

quadratic characteristic Wu

$$df(x,y) = d_x f(x,y) + \omega(x) d_y(f,y)$$

exterior simplicial derivative on

$\Lambda = \{f : \{(x,y) \in G \times G, x \cap y \in G\} \to \mathbb{R}\}$

$D = d + d^*$ Dirac

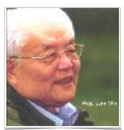
$H = D^2 = dd^* + d^*d$ Hodge

$b_k = \dim(\ker(H_k))$ Betti

We get analog stories to Euler case:

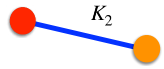
Euler-Poincare | Gauss-Bonnet | Brouwer-Lefschetz | Poincaré-Hopf

Leonhard Euler 1707-1783



Wenjun Wu 1919-2017

x	x	$\omega(x)$
●	{1}	1
●	{2}	1
—	{1,2}	-1

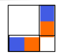
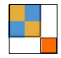


$G = \{\{1\},\{2\},\{1,2\}\}$

$b = (1,0)$ Betti vector

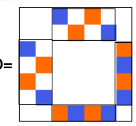
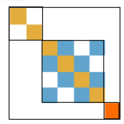
$\chi(G) = 1$

constant 0-forms are harmonic

D=  H= 

(x,y)	x	y	$\omega(x)\omega(y)$
●●	{1}	{1}	1
●●	{2}	{2}	1
● —	{1}	{1,2}	-1
● —	{2}	{1,2}	-1
— ●	{1,2}	{1}	-1
— ●	{1,2}	{2}	-1
=	{1,2}	{1,2}	1

$\chi_2(G) = -1$

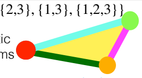
D=  H= 

$b = (0,1,0)$ quadratic Betti vector

constant 1-forms are harmonic

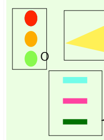
$G = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,3\},\{1,2,3\}\}$

the only harmonic quadratic differential forms are 2-forms



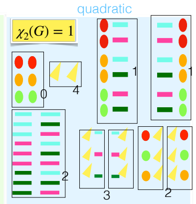
linear

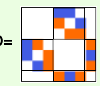
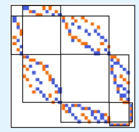
$\chi(G) = 1$

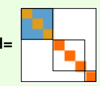
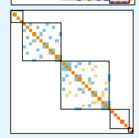


quadratic

$\chi_2(G) = 1$



D=  D= 

H=  H= 

Λ is 7-dimensional b=(1,0,0)

Λ is 37-dimensional b=(0,0,1,0,0)

Valuations

Hadwiger 1908-1981

Dan Klein

J-C. Rota

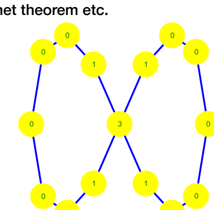
Basis for linear valuations : f_k

Valuations are q+1 dimensional

Basis for quadratic valuations : f_{kl}

Valuations are (q+1)(q+2)/2 dimensional

Any valuation comes with a Gauss-Bonnet theorem etc.

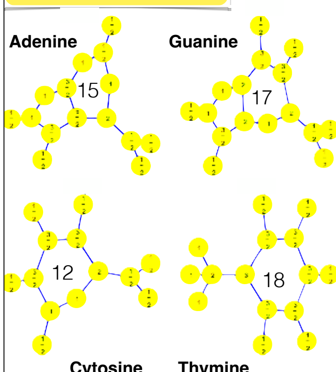


Gauss-Bonnet

$$K(v) = \sum_{x \sim y, v \in x} \frac{\omega(x)\omega(y)}{|x|}$$ curvature

$$\sum_{v \in V} K(v) = \chi_2(G)$$

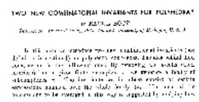
Gauss-Bonnet for quadratic characteristic



Topological

Quadratic cohomology is an example of of a topological invariant, not homotopy invariant!

Bott Introduced term "Topological Invariant"

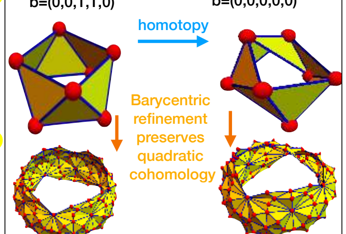


1923-2005

Example:

Cylinder b=(0,0,1,1,0)

Möbius strip b=(0,0,0,0,0)



b=(0,0,1,1,0) b=(0,0,0,0,0)

Boundary Formula

Let G be a q-manifold with boundary

$$\chi_2(G) = \chi(G) - \chi(\delta G) = \chi(\text{int}(G))$$

Proof: a manifold with boundary is characterized by having every U(x) for x in the interior to have a sphere boundary S(x) so that

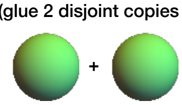
$\chi_2(U(x)) = \chi_2(B(x)) - \chi_2(S(x)) = \omega(x)$

which agrees with $\chi(U(x))$. Therefore, using inclusion exclusion, the linear and quadratic characteristic agree on open sets.

q even $\chi_2(G) = \chi(G)$

q odd $\chi_2(G) = -\dfrac{\chi(\delta(G)}{2}$

(glue 2 disjoint copies of G along δG)



\+ = 3 sphere

Cubic Cohomology

G finite abstract simplicial complex.

$$\chi_3(G) = \sum_{x \cap y \cap z \in G} \omega(x)\omega(y)\omega(z)$$

cubic characteristic

$$df(x,y,z) = d_x f(x,y,z) + \omega(x) d_y(f,y,z) + \omega(x)\omega(y) d_z(f,y,z)$$

exterior simplicial derivative on

$D = d + d^*$ Dirac

$H = D^2 = dd^* + d^*d$ Hodge

$b_k = \dim(\ker(H_k))$ Betti

Let G be a q-manifold with boundary

$$\chi_3(G) = \chi(G) + \chi(\delta G)$$

All balls have Cubic characteristic 1

q even $\chi_3(G) = \chi(G)$

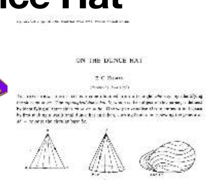
q odd $\chi_3(G) = \dfrac{\chi(\delta(G)}{2}$

Complete Complexes

Complex	Euler	Betti	Wu	Betti	Cubic	Betti
K1	1	(1)	1	(1)	1	(1)
K2	1	(1,0)	-1	(0,1,0)	1	(0,0,1,0)
K3	1	(1,0,0)	1	(0,0,1,0,0)	1	(0,0,0,0,1,0,0)

Dunce Hat

homotopic to 1 not contractible



$\chi_1(G) = 1$ $b_1(G) = (1,0,0)$

$\chi_2(G) = 1$ $b_2(G) = (0,0,0,1,2)$

$\chi_3(G) = 1$ $b_2(G) = (0,0,0,0,0,1,2)$

higher cohomology sees non-contractibility

Unit 12: Higher Green

The k-point **Green function** $g_m(x_1,\dots,x_k) = \prod_{j=1}^k \omega(x_j)\chi_m(\bigcap_{j=1}^k U(x_j))$ for the **m'th characteristic** $\chi_m(A) = \sum_{x \in A^m, \bigcap_j x_j \in A} \prod_{j=1}^m \omega(x_j)$ of $A \subset G$ satisfies $\chi_m(G) = \sum_{x \in G^k} g_m(x_1,\dots,x_k)$.

12.85. The **m'th characteristic** is defined as $\chi_m(G) = \sum_{\mathbf{x} \in G^m, \bigcap \mathbf{x} \in G} \omega(x)$. It is the Euler characteristic in the linear case $m = 1$ and the quadratic characteristic in the quadratic case $m = 2$. If used vector notation $\mathbf{x} = (x_1,\dots,x_m)$ and multi-index notation $\omega(\mathbf{x}) = \prod_{j=1}^m \omega(x_j)$ and $\bigcap \mathbf{x} = x_1 \cap x_2 \cap \cdots \cap x_m$.

12.86. The **k-point Green function** on G^k is $g_m(\mathbf{x}) = \omega(\mathbf{x})\chi_m(\bigcap U(\mathbf{x}))$, where $U(x_j)$ is the star of x_j and the star $U(\mathbf{x})$ of $\mathbf{x} \in G^k$ is the intersection of the stars of $x_1,\dots,x_k$. The k-tuple $\mathbf{x} = (x_1,\dots,x_k)$ does not need to intersect but the k-point Green function are still local because $U(\mathbf{x})$ is empty, if one of the "particles" x_j is out of reach.

12.87. The Green function identity $\chi(G) = \sum_{x,y} g(x,y)$ for the quadratic case $m = 2$ generalizes but already in the cubic case $g(x,y,z)$ is a tensor. For $m = 2$, we could think of $g(x,y)$ as a matrix, where it was the inverse of $L(x,y)$. While $\chi_m(G)$ involves only m-tuples of G that intersect, the Green functions $g(x_1,\dots,x_k)$ involve all k-tuples of G. Think of a **k-point potential energy**. The energy theorem generalizes to

Theorem: $\chi_m(G) = \sum_{\mathbf{x} \in G^k} g_m(\mathbf{x})$.

Proof. Generalize to energized complexes where $\omega(\mathbf{x})$ is replaced with a general function $h_m(x_1,\dots,x_m)$ of m variables producing a m-linear valuation. Since $h_m \to g_k$ is linear, we only need to verify the statement in the simplest possible case, where h is 1 only for a single configuration $\mathbf{z} = (z_1,\dots,z_m)$ and 0 else. The left hand side is then 1. On the right hand side, we have to look at all $\mathbf{x} = (x_1,\dots,x_k)$ for which $\mathbf{z} \in U(\mathbf{x}) = U(x_1) \cap U(x_2) \cap \cdots \cap U(x_m)$. $\sum_{\mathbf{x} \in G^k} \omega(\mathbf{x})\chi_m(U(\mathbf{x})) = \prod_{j=1}^k \sum_{x_j \in G} \omega(x_j)\chi_m(U(x_j)) = 1$. $\square$

12.88. A second major point is the **sphere formula** for $S(\mathbf{x}) = \delta U(\mathbf{x}) = B(\mathbf{x}) \setminus \mathbf{U}(\mathbf{x})$, where we write similarly as $U(\mathbf{x}) = \bigcap_j U(x_j)$ also $B(\mathbf{x}) = \bigcap_j B(x_j)$.

Theorem: $\sum_{\mathbf{x} \in G^k} \omega(\mathbf{x})\chi_m(S(\mathbf{x})) = 0$.

12.89. This means that the total m'th characteristic of all unit spheres of even k-point configurations is the same than the total m'th characteristic of all unit spheres of odd k-point configurations.

Proof. The energy theorem also works of $U(\mathbf{x})$ is replaced by $B(\mathbf{x})$, which is the closure of $U(\mathbf{x})$. The two equations $0 = \sum_{\mathbf{x} \in G^k} \omega(\mathbf{x})\chi_m(U(\mathbf{x}))$ and $0 = \sum_{\mathbf{x} \in G^k} \omega(\mathbf{x})\chi_m(B(\mathbf{x}))$ and the local valuation formula blow prove the theorem. $\square$

12.90. For characteristic $m = 1$, the formula $\chi(\overline{U}) = \chi(U) + \chi(\delta U)$ follows from the fact that every simplex in $\overline{U}$ is either in U or δU. The **local valuation formula** is:

Theorem: $\chi_m(B(\mathbf{x})) = \chi_m(U(\mathbf{x})) - (-1)^m \chi_m(S(\mathbf{x}))$.

12.91. For odd m this means $\chi(U) + \chi(\delta U) = \chi(\overline{U})$. For quadratic characteristic $m = 2$, one has $\chi_2(U(x)) - \chi_2(S(x)) = \chi_2(B(x))$. If $B(x)$ is a replaced with a manifold G with boundary and we chose $k = 1$, one gets the boundary formula $\chi_2(G) = \chi_2(\text{int}(G)) - \chi_2(\delta G)$ which then simplifies to $\chi_2(G) = \chi(G) - \chi(\delta G)$.

Theorem: If G is manifold we have $\chi_m(G) = \chi(G) - (-1)^m \chi(\delta G)$.

Proof. Use induction and that the left hand side is $\chi_m(\text{int}(G)) - (-1)^m \chi_m(S(\delta G))$. □

12.92. Higher characteristics of manifolds with boundary can be expressed using Euler characteristic. For even-dimensional manifolds with boundary $\chi_m(G) = \chi(G)$. For odd-dimensional manifolds with boundary $\chi_m(G) = -(-1)^m \chi(\delta G)/2$. For even dimensional balls G for example, where the boundary is an odd-dimensional sphere, $\chi_m(G) = 1$ independent of m, while for odd-dimensional balls $\chi_m(G) = (-1)^{m+1}$.

12.93. Higher characteristic $\chi_m(G)$ can be written as $\chi_m(G, G, \dots, G)$ if one defines the multi-linear expressions $\chi_m(A_1, \dots, A_m) = \sum_{x_1 \cap \dots \cap x_m \in G, x_i \in A_i} \omega(\mathbf{x})$. Like for multi-linear forms in linear algebra, fixing all but one coordinates produces a valuation. Like Euler characteristic had the f-vector and quadratic valuations had the f-matrix f_{kl}, multi-linear forms define the **f-tensor** $f_{k_1,\dots,k_m}$. Applied to m Dehn-Sommerville vectors, we get Dehn-Sommerville invariants for manifolds answering a question of Gruenbaum. The most prominent Dehn-Sommerville invariant is $X = (1, -1, 1, -1, \dots)$ which produces characteristic invariants $f(X, X, \dots, X)$. Let us abbreviate m-valuation for multi-linear valuation of m variables:

Theorem: χ_m is the only m-valuation that is Barycentric invariant.

Proof. $(1, -1, 1, -1, \dots, \pm 1)$ is the only eigenvector of the **Barycentric operator**. □

Quantities with this property are **topological invariants**.

Higher Green

G finite abstract simplicial complex.

$$\chi_m(G) = \sum_{\mathbf{x} \in G^m, \bigcap \mathbf{x} \in G} \omega(\mathbf{x})$$

m-th characteristic. Total m-energy

$$g_m(\mathbf{x}) = \omega(\mathbf{x})\chi_m(U(\mathbf{x}))$$

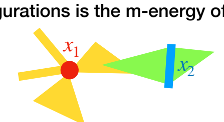
k point Green function

energy of a k-particle configuration.

Theorem: for all G, for all k and m

$$\chi_m(G) = \sum_{\mathbf{x} \in G^k} g_m(\mathbf{x})$$

"Total m-energy of all k-particle configurations is the m-energy of G"



$g_m(x_1, x_2) = \omega(x_1)\omega(x_2)\chi_m(U(x_1) \cap U(x_2))$

Notation

G finite abstract simplicial complex.

$\mathbf{x} = (x_1, \dots, x_k)$ k tuple of particles in G

$\mathbf{x} = (x_1, \dots, x_m)$ m- molecule if $\bigcap x \neq 0$

(order of particles do not matter)

$$\omega(\mathbf{x}) = \omega(x_1)\cdots\omega(x_m)$$

energy of a tuple of simplices

$$\bigcap \mathbf{x} = \bigcap_{j=1}^{m} x_j$$

simultaneous intersection

$$U(\mathbf{x}) = \bigcap_{j=1}^{k} U(x_j)$$

star of a k-particle configuration

m energy level

k number of particles

These are independent numbers

The story is Bosonic and force related

Case m=1:

1-point functions : "vacuum expectations"

$$\chi(G) = \sum_{x \in G} \omega(x)$$

Euler Characteristic

$$g(x) = \omega(x)\chi(U(x))$$

curvature or index, dual to $\omega(x)$

$$\chi(G) = \sum_{x \in G} \omega(x)\chi(U(x))$$

a Gauss Bonnet result

super trace of 2-point Green function. Using the valuation property and the definition of $\chi(G)$, this gives:

$$0 = \sum_{x \in G} \omega(x)\chi(S(x))$$

sphere theorem

Case m=2:

2-point functions g(x,y)
"potential energy"
"probability amplitude"

$$\chi_2(G) = \sum_{x,y \in G, x \cap y \in G} \omega(x)\omega(y)$$

Wu Characteristic

$$g(x,y) = \omega(x)\omega(y)\chi(U(x) \cap U(y))$$

Green functions

$$\chi_2(G) = \sum_{x,y \in G} g(x,y)$$

energy theorem

$$0 = \sum_{x,y \in G} \omega(x)\omega(y)\chi(S(x) \cap S(y))$$

quadratic sphere theorem

Proof: generalize!

The result works for any multi-linear valuation.
energize the simplices arbitrarily

$$\chi_m(G) = \sum_{\mathbf{x} \in G^m, \bigcap \mathbf{x} \in G} h(x_1, \dots, x_m)$$

The map $h \to g_k(h)$ is linear.
Verify for basis.

Local Valuation

$$\chi_m(B(x)) = \chi_m(U(x)) - (-1)^m \chi_m(S(x))$$

Sphere formula

For all positive k and m and all G

$$\sum_{\mathbf{x} \in G^k} \omega(\mathbf{x})\chi_m(S(\mathbf{x})) = 0$$

Manifolds

For all q-manifolds with boundary

$$\chi_m(G) = \chi(G) - (-1)^m \chi(\delta G)$$

it can be seen as a generalization of

$$\chi_m(B(x)) = \chi_m(U(x)) - (-1)^m \chi_m(S(x))$$

using that Barycentric refinement reserves higher characteristic:

$$\chi_m(G) = \chi_m(G_1)$$

For even dimensional manifolds:

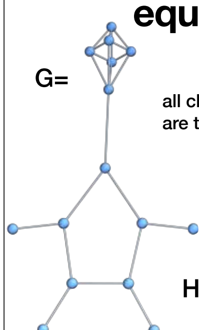
$$\chi_m(G) = \chi(G)$$

For odd-dimensional manifolds:

$$\chi_m(G) = -(-1)^m \chi(\delta G)/2$$

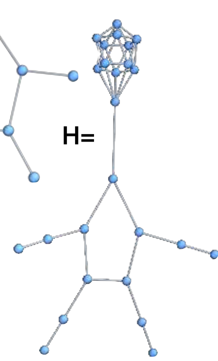
Topological equivalent

G=



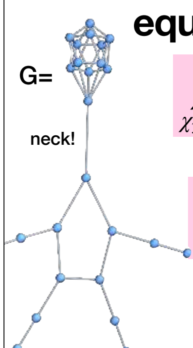
all characteristics are the same.

H=



Not topological equivalent

G=

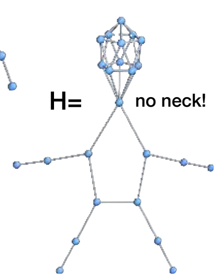


neck!

$\chi(G) = 1$
$\chi_2(G) = 11$
$\chi_3(G) = -29$

$\chi(H) = 1$
$\chi_2(H) = 9$
$\chi_3(H) = -23$

H=



no neck!

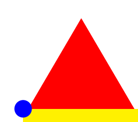
This is the end of book I of

ELEMENTS OF FINITE GEOMETRY

WITH COLOURED DIAGRAMS AND SYMBOLS
ART MEETS SCIENCE EDITION

BY OLIVER KNILL

SURVEYOR OF IMAGINARY MATHEMATICAL ISLANDS AND AUTHOR OF NUMEROUS MATHEMATICAL WORKS



Footnotes

This section contains some remarks to the text. Brevity in the main text was not to get bogged down. There were previous attempts to summarize things and they always hit a wall. I gave myself a self-imposed limit of 2 pages per unit and a one minute limit for the accompanying videos. The illustrations are from slides written for such youtube shorts written during the summer. Texts were drafted partly during the earlier summer during travels. It started in the Swiss alps with a view onto the Mischabel mountain range seen on the title page. One can see about 12 mountain peaks. The footnotes and references aim less at optimizing length and therefore meander more. Possible in the future could be an appendix with examples and exercises. But it will be more important to work on the next batch of 12 units, if circumstances permit. This summer was the first in 24 years, where I had not been teaching in summer school, providing an opportunity to work on this project.

0. Introduction

The motivation on why to focus on finite structures is a reaction to the incompleteness theorem [53] which stresses that the foundations of mathematics will always remain uncertain if the assumptions are strong enough. Popularized masterfully in [64], it had been one of the most pivotal high school experiences that drew me personally to mathematics. I got exposed to "strict finitism" by Erwin Engeler at ETH (he taught us a set theoretical "topology" course and "mathematical software" course). He mentioned strict finitism once in person but [43] is one of Engeler's articles from 1971. Ernst Specker (from whom I took two semesters of linear algebra, as well as logic and model theory courses and also participated in various seminars), mused about the relation between finite mathematics and the continuum and informed the class once about the work of Paul Kustaanheimo [139], an attempt to do mathematics in finite fields, rather than the real numbers. Kustaanheimo collaborated with Eduard Stiefel while visiting Zuerich in 1964, certainly knowing Specker, as Specker and Stiefel both were students of Heinz Hopf. Peter Läuchli, a student of Specker (from whom I took two semesters of "calculus" (both single and multi-variable) and a logic course about forcing) gave later a mesmerizing course on non-standard analysis based on Nelson's internal set theory [156, 157]. See [98] for an other personal story about Specker and Läuchli (and other ETH teachers). The textbook that Läuchli recommended was [172] and he followed it partly. IST an effective link between finitist and traditional mathematics as it just changes language and not structures. It shows that compact spaces can be treated like finite spaces. Nelson also reformulated probability theory [157] and called it "radically elementary". Finite models were also pushed by other of my teachers. Jürgen Moser mentioned that finite systems show similar KAM features than the continuum [168] and my PhD advisor Oscar Lanford II worked in his later career on finite approximations of dynamical systems [141]. Finally, finitism is a honest reaction to the fact that all data we ever acquire and process are finite. It is also a computer science approach to mathematics. The Taylor formula mentioned in the introduction appeared in a Pecha-Kucha talk [97], a presentation in which 20 slides are shown with a strict 20 second limit to talk for each. The discrete Taylor theorem is attributed to James Gregory, a contemporary of Newton. One can find discrete calculus also in the original writing of Leibniz. I used discrete calculus excessively in single variable Math 1a courses from 2011 to 2024. I was fortunate to have been given the freedom to experiment with a fresh take on single variable calculus then. [190] contains the formula on page 69 in the context of interpolation. The topics discussed in these notes (and beyond) have previously seen attempts to be summarized: [92, 94, 96, 108, 133, 134, 135].

1. Gauss-Bonnet

For the discrete Gauss-Bonnet theorem, see [39, 144, 87, 113, 86, 101, 107]. Norman Levitt, was the first who has introduced curvature of complexes in higher dimensions. He did not stress it as a Gauss-Bonnet theorem. The integral geometric interpretation as an index expectation verifies with Nash's embedding theorem that the Gauss-Bonnet-Chern curvature for even dimensional manifolds is a limiting case of this discrete measure. Many other curvatures have been proposed graph theory, some using probability theory [161, 76]. For me, a curvature is interesting if it satisfies Gauss-Bonnet. Also notions like sectional curvature should be curvatures of a two dimensional geodesic surface passing through a point at which the curvature is measured. I learned about Levitt's paper only years after [86]. I had seen the general formula only after painfully going through small dimensional manifold cases, where Dehn-Sommerville relations obscure a bit the patterns. This already happens in the case of oldest formula $K = 1 - \deg(v)/6$ for triangulated two dimensional surfaces, where the Dehn-Sommerville identity $2|E| = 3|F|$ hides that $K = 1 - \deg(v)/2 + \deg(v)/3$. This curvature formula appears in [42] after writing [87]. A crucial problem had been for then to explain why we always see constant zero curvature for odd-dimensional manifolds. This turned out to be a rich problem. It motivated to look at integral geometric approaches and a produced several encounters with the Dehn-Sommerville story. The integral geometric connection proves that the curvature is the real deal. The discrete curvature is not just some "analogy". It produces the Gauss-Bonnet-Chern integrand if we take a discrete approximation of a compact Riemannian manifold. Related more general curvatures appeared in [63, 131]. For the classical case, see [196]. The classical Gauss-Bonnet-Chern evolved over decades: [65, 6, 45, 5, 24, 25, 30] Discretisations of space have appeared long before Poincaré, but less for general manifolds. Historically, one has focussed more on polytopes [199, 58]. "Monsters" as Lakatos described counter examples to the Gem formula have made geometers more careful. It is a "delicate task" [36]. The result was that discussions about polyhedra are mostly limited to convex objects. If we discard the continuum, we need to be extra clear, in order not to repeat the scandal which Lakatos described masterfully. If we have a finite combinatorial object without any embedding into some Euclidean space, we have to be precise with what is considered a "face" of a given dimension. For history see [38]. For discrete geometries see [35, 169, 153] for computational geometry [36, 11], for integrable systems [16], for computer graphics or computational-numerical methods [32, 17, 162]. For graph theory in general see [37, 60, 150, 7] For combinatorics of simplicial complexes in general [147, 186, 170].

2. Poincaré-Hopf

The discrete Poincaré-Hopf appeared first in [88]. It was useful for [121]. The work with Frank Josellis in [75] led to simplifications. We hope to get to Morse and Lusternik-Schnirelmann topics in the second volume. The parametrized version appeared in [116]. More elaborations came in [115] and [117]. A poetic approach to numbers is obtained by looking at numbers geometrically [103]. I had then not been aware of [14] which introduced that complex already. [103] however allows to interpret the Riemann hypothesis (the Mertens connection) from a topological point of view. For an early discrete version in two dimensions based on an embedding in a surfaces, see [52]. I learned about Glass'theorem from Kate Perkins thesis at Harvey Mudd [164]. The speed with which the simplex generating function can be computed recursively using the formula given in the text, needs to be studied more. Computing the number of simplices in a graph is known to be a hard problem. The clique problem is NP complete [77]. What the recursive formula does is relegate the problem to subgraphs in unit spheres. To the classical story of Poincaré-Hopf: Poincaré proved the index theorem in chapter VIII of [165]. Hopf extended it to arbitrary dimensions in [66]. For manifolds with boundary, see [167]. [185] contains a modern exposition. The collection [71] contains in particular an article on the history of graph theory. A story about Euler characteristic and polyhedra is told in [171, 28]. See [61, 184, 174] for notations in algebraic topology, [60, 13, 18] for graph theory. Valuations have been pioneered by Klee [82] and Rota [173] who already saw that there is a unique normalized valuation. As for Hadwiger see [59, 80].

3. Index expectation

The Buffon needle problem, stated and solved in 1777, is one of the first problems in the theory of geometric probability. The subject is also called integral geometry. $L = \pi \mathrm{E}[n]$ gives the length if lines are distance 2 apart. See [176]. Crofton's book on local probability is [29]. More general references on integral geometry are [175, 81, 180, 15, 158]. The first time, it seems that curvature was seen as an expectation of indices is in the work of Banchoff [8, 9]. It has also appeared in graph theory [10]. As for index expectation, see [90] [89], [111] [124] [125]. For the Nash embedding theorem source, see [138]. Gauss-Bonnet-Chern is older than Chern, if one allows embeddings. The first intrinsic proof of GBC without embedding the manifold is [24]. For historical remarks see [25]. I myself learned GBC first from Patodi's proof [30]. It was Konrad Osterwalder who made me look into chapter 12 of the book of Cycon-Froese-Kirsch and Simon to assist in a seminar on Patodi's proof of the Gauss-Bonnet-Chern theorem. I spent a week alone on the alp “Salmenfee” in the Swiss mountains during a summer break, working through that chapter. The seminar eventually did not take place; the exposure to that book however turned out to be life changing for me, as the book contained a nice concise introduction into modern Riemannian geometry and interesting spectral theory. Integral geometry is extremely useful as it links the discrete with the continuum. Because Poincaré-Hopf indices are divisors, there is a direct link between the discrete and continuum. Integral geometry allows to bypass any discrete Riemannian manifold set-ups. Distances can be implemented using embeddings for example. Index expectation was crucial for the product formula for the Shannon product [183]. See also [119].

4. Euler's Gem

The task to define what a manifold is combinatorially essentially boils down to the task to define "spheres", a question going back to Herman Weyl [193], at a time when the "Grundlagen crisis" had been raging still. (There is still a Grundlagen crisis today of course but similarly as the possibilities of a nuclear armagedon danger is blended out from the public consciousness, mathematicians usually ignore the possibility of an inconsistency to occur. They have doubled down even by using structures that go beyond ZFC like Grothendieck universes to avoid objects like the category of all categories. There are two important early approaches to discrete manifolds. For simplicial complexes, see [74, 186, 187]. One approach is by Robin Forman [47, 48, 49, 50] who built "discrete Morse theory", the other by Ivashchenko (Evako) [69, 70, 23] whose work is sometimes placed into a field called "digital topology". Forman defines spheres Morse theoretically using functions, Evako uses induction. The compact version done here has been developed by us, first in the graph theoretical set-up, later in abstract simplicial complexes using the Alexandroff topology [2, 148, 128]. What is important to make the definition decidable is to use "contractibility" (sometimes called "collapsible") and not "homotopic to 1" in the definition of a sphere. With the later, one would run into computer science trouble: there would be no Turing machine which could decide whether a given structure is a manifold or not. With "contractibility" as defined, there is a definite algorithm which terminates polynomial time once the dimension is fixed and decides whether a given graph or a given finite abstract simplicial complex is a manifold or not. One of the simplest complexes that is homotopic to 1 but not contractible is the dunce hat [198]. Discrete Reeb [118] shows equivalence of Morse and Recursive approaches. A good general reference for the topic of Euler's gem is [171]. I gave a mathtable talk about this on 2/6/2018 to honor Euler's day 2/7/18 [109] The innocent looking Euler Gem formula is infamous for having been proven wrong repetitively, and not by amateur mathematicians or but by the best mathematicians in the world. It so has become a show case for the epistemology of mathematical discovery [140]. I myself disagree with Lakatos that science is an evolutionary approach to truth. It should be true from the beginning. The culprit usually is that the definitions are not good. The algebra of the join operation goes back to [200] in graph theory. It might even have appeared already in the Principia Mathematica from 1910. The analog for simplicial complexes is to take $A \cup B \cup \{a \cup b, a \in A, b \in B\}$ for the join of two closed sets. It produces a monoid structure on complexes and spheres are a sub-monoid. Also Dehn-Sommerville spaces form a submonoid. The classification of regular polytopes is usually done in the continuum in the context of polytopes [26, 73, 188, 199, 58], which invoke Euclidean space and convexity. [57] discussed the difficulty with the term "polytop". For [28],a polytop is defined as as a convex body with polygonal faces. For source work, [179, 182, 166]. The origin of Euler characteristic is captivating [1]. Polytop definitions are given in [179, 28, 58, 73]. Topologists started with new definitions [4, 46, 27, 184]. Abstract simplicial complexes appeared in 1907 [31]. See [22, 154]. [4] uses the name unrestricted skeleton complex. In [194] there are called symbolic complexes. For references on simplicial complexes, see [74, 186, 187] but definitions can vary. Especially the treatment of $\emptyset$. Insisting on having simplicial complexes not contain $\emptyset$ is much, much more natural and makes the theory look like the theory we all know when studying differential topology or differential geometry.

5. Euler-Poincare

Finite exterior calculus has been used already by the founders of algebraic topology. The one-dimensional discrete Laplacian H_0 is now called the **Kirchhoff matrix** [78]. Discrete Hodge theory appeared first in [40]. One of the talks given by Beno Eckmann at ETH given at the University of Zürich that focussed on Euler characteristic made me personlly curious about this topic. Looking at higher order Laplacians is pretty established also in the discrete (an example [68]). Discrete calculus has been re-invented again and again especially in the applied and computer science literature. Examples are [33, 34]. Discrete McKean-Singer [149] appeared in [91]. Taking simple ideas going back to Dirac and Hodge (they were office neighbors who did not talk much together as Michael Atiyah once happily pointed out in an interview as this resulted him building a career combining both their ideas). Discrete calculus has been reinvented again and again, in later time, it has appeared mostly in applied or computer science literature. [41]. The main definition of using linear algebra in topology have already been forged by Poincaré or Betti. The matrix D is usually given in terms of its blocks, which are called **incidence matrices**. It would have been so much easier already at the time of Poincaré to combine these matrices to a common matrix d on all forms and to define $D = d + d^*$ and then study the matrix D^2. See [94]. The insight that one does not have to look at anti-symmetric functions, but just can look at functions, once an orientation is fixed on simplices, is probably due to Hassler Whitney who used this especially in the context of the cup product. To bypass the chain complex notations and use linear algebra is due to Hodge. It is such a relieve especially when working with computer algebra implementations not to have to deal with equivalence classes, but simply with null-spaces of matrices. The concept of the square root of the Laplacian is due to Dirac. Clifford algebra constructions show that we can bypass awkward Dirac matrix constructions. The Hodge definition for cohomology is more intuitive than equivalence classes "closed forms modulo exact forms". It is a good (a bit challenging but doable) exercise for an introduction linear algebra course to show that the two pictures equivalent. About the notation, we use H for Hodge and because we want to use L for the connection Laplacian. We write **Dirac operator** and not **Dirac matrix** in order not to confuse with Dirac matrices in mathematical physics, which are used to get the square root of the Laplacian.

6. Unimodularity

The result was discovered in December 2015 while experimenting with quadratic cohomology. Neither any discrete nor continuum analog of this story seems have appeared before. It was presented on a mathtable talk "Bowen-Lanford Zeta function" on 10/18/2016 [100], referring to [19]. The proof appeared in [102]. Some extension appear in [112] like when $\omega(x)$ is replaced with 1. A reason for the delay was a summer excursion into number theory [104]. The result still surprises me today. A more direct alternative proof of unimodularity was given in [152]. I still feel the unimodularity result is remarkable and I'm glad I spent some effort to have it published [121]. Polishing something for an actual publication uses a lot of time and often is a fool's errand

anyway, if one deals with a topic, that has not been covered before. There were various generalizations [122], [114], [120], and [123]. For some spectral problems in the context of Dirac and connection matrices, see [136].
The connection story is related to the **Szpilrajn-Marczewski theorem**: any finite simple graph A can be realized as a connection graph of a finite set G of non-empty sets [189, 163] as pointed out in [119].

7. Lefschetz fixed point formula

The discrete Brouwer Lefschetz theorem appeared in [93]. My proof there was close to Hopf's approach [67] which also is in [4] (Alexandrov and Hopf were friends who had collaborated extensively like in [3]. Textbooks cover it in [72, 184, 55, 54]. A version of the fixed point theorem in dimension 1 is the **Nowakowski-Rival fixed edge theorem** [160] and a generalization [44] to a commutative family of such maps. For the classical Lefschetz fixed point theorem and Brouwer fixed point theorem see [143, 21]. The Brouwer fixed point theorem is usually stated for balls in Euclidean space, where one can prove it also with discrete methods, like Gale's proof using q-dimensional Hex games. The text mentions some work on discrete Hurwitz [137] which I discussed in 2012/2013 with Thomas Tucker. I talked about this a couple of times, like at [96] or on July 7, 2026 in a youtube presentation. One reason why not pursue this more was when realizing that the result is a rather direct consequence of the Burnside lemma applied to each dimension sector of the simplicial complex. We hope to mention it more in a later volume of this project.

8. Sphere formula

The sphere formula is also a show case how to go from graphs to simplicial complexes. In both cases there are "unit spheres". For graphs, it is the graph generated by the neighbors. For simplicial complexes, the unit spheres are the boundaries of the smallest open sets. For traditional Euler characteristic, the sphere formula follows from the Green formula $\chi(G) = \sum_x \omega(x)\chi(U(x))$ which is $\mathrm{str}(L)$. One also has $\chi(B(x)) = 1$ so that $\chi(G) = \sum_x \omega(x)\chi(B(x))$ Together with the definition $\chi(G) = \sum_x \omega(x)$ this proves the sphere formula. The sphere formula immediately shows that odd dimensional manifold have Euler characteristic zero. [121, 129]. The Hydrogen relation was also discussed in [107] and [110] where it is shown that $g = \log(f)$ has the property that $g'(1)$ is the average simplex cardinality. For Dehn-Sommerville, see [83, 159, 155, 145, 20, 62, 79, 12]. Why unit spheres or hyperbolicity in simplicial complexes are less studied is maybe also due to notation. I myself got motivated from dynamical systems, especially classes of Lanford or [142] or [197], the Hillerod conference I attended, as well as attemps of mine to understand hyperbolicity in ergodic set-ups [84, 85]. Traditionally, one has focused on the vertices of a simplicial complex rather than all simplices. [191] defined a **neighborhood** of a k-simplex, still using geometric realizations. Veblen also used already the terminology of "stars". The notion of "Stars" was established in combinatorial topology like [2] . It also entered some calculus textbooks like Whitney [195]. But even Alexandroff look at it primarily at geometric realizations of cell complexes. For the unit sphere $S(v)$ of a zero dimensional complex, there is no interesting hyperbolicity to be observed. There are strange names also used in the literature for $S(v)$.

One calls it a "link". Maybe, one should credit the discrete Morse theory of Forman for centering on simplices as "points". This is also the point of view for Barycentric refinement. If G is a complex, then the Barycentric refinement of G has the simplices of G as "points". The Barycentric refinement, then formalizes this and has the original simplices now zero dimensional objects. Most of the literature still looks at Barycentric refinements not combinatorially but by using a geometric realization of the complex in an Euclidean space.

9. Level sets

Much of calculus or algebraic geometry or differential topology deals with level sets of functions. Arthur Sard [177, 178] and Anthony Morse [151] established that a smooth map f from a manifold M to a manifold N has the property that for almost all $y \in \text{ran}(f)$, the level set $\{x \in M, f(x) = y\}$ is a manifold. Already simple examples of polynomial maps like $x^2 - y^2 = 0$ show that level sets are in general not manifolds. It can come a bit as a shock that in the discrete singularities do not occur. Level sets of any function always are manifolds if not empty. Of course, we need to define level set properly. This was done in [99] in 2015. It was generalized in 2024 [132]. Working with Jennifer Gao who wrote a thesis in that area helped a lot during this time [51]. The level set result get for teaching purposes in Math 136 differential geometry classes. It is simple enough to be squeezed into a discussion of what a manifold is. In [134], it was generalized to Dehn-Sommerville manifolds, an approach to manifolds which is recursive but only makes an assumption about Euler characteristics. For the history of the development of the classical notions of manifolds, see [71, 181, 146].

10. Index formula

The formula first appeared in [89] in 2012 and [95] in 2013. There was no level set construction yet and things were more complicated and ugly from a current perspective. It was an application for [90] but it required to define a level set structure defined by a function. It was valuable time however to work on this because it enabled the level set result to appear three years later. I like the index formula because it is maybe the swiftest way to see why odd-dimensional manifolds are flat. This observation was one of the earliest I made and it bothered me in 2010 that while possible to verify it for 3-manifolds, I had been unable to verify it for 5 manifolds already. It is also intuitive to see curvature as a genus expectation of random sub-manifolds in unit spheres. This shows that if we take a 4 manifold, a small sphere $S_r(x)$ in it and a random function, then the expected genus of contour surfaces $\{f(y) = f(x)\}$ is curvature. Since this is related to curvature of random 2-manifolds, this is conceptionally closely related to scalar curvature which is the expected sectional curvature at a point. It motivates to look at Euler characteristic as a serious functional with possible physical relevance. The index formula is non-trivial already for 2-manifolds G. Put random function values on the vertices of a 2-manifold and at each node v look at the average number $X(v)$ of sign changes of $f(w) - f(v)$ when w goes along the circle around v. Then $K(v) = 1 - \text{E}[X(v)]/2$ is equal to $1 - d(v)/6$, the Eberhard curvature.

11. Quadratic cohomology

This started in [101] in 2016 and cohomology in [106], where the terminology "Wu cohomology" or "connection cohomology" was used. Wen-Tsun Wu in 1953 first considered this multi-linear valuation in 1953 [192]. [56] picked it up and conjectured the existence of Dehn-Sommerville invariants, which was confirmed in [101]. Tamas Reti pointed out in 2016 a relation to the **Zagreb index**. When the fusion inequality for simplicial cohomology [127] was generalized to higher cohomology, [130], the name "quadratic cohomology" stuck. We mentioned the theory last in [136] in the context of Birkhoff's fixed point theorem. I gave a table talk on March 8, 2016 about Wu characteristic. [105]. It needs to be studied much more especially also in the context of the fusion story [127, 130], where the cohomology is used also for open sets and interface cohomologies between an open and closed set are used. One can look at the quadratic cohomology for the complement of a path in a 3-sphere for example. Does this give interesting knot invariants? In [106] we computed several examples. An interesting one is the dunce hat because its quadratic cohomology is different from the quadratic cohomology of a point. It demonstrates that quadratic cohomology is not a homotopy invariant.

12. Higher Green

This section was covered [126], work done mostly in the winter break of 2022, when visiting Burlington in Vermont. It generalizes the energy theorem in the case when $k = 2$ and $m = 1$, the Euler characteristic case. The case $k = 1$ has appeared as $\mathrm{str}(g) = \chi(G)$. It is remarkable, how prominently Green functions appear in physics. One usually associates Green functions with the inverse of Laplacians but this is only the 2-particle story. In the continuum, it leads to technical difficulties as the Laplacians are not invertible. Particle interaction potentials have singularities. The Green function of the Laplacian in Euclidean space $\mathbb{R}^3$ for example leads to the $1/r$ potential energy, we see in physics, like in electro magnetism or gravity. Every geometric space defines so a potential, in $\mathbb{R}^2$ it would be log in $\mathbb{R}^4$, two celestial bodies would interact with a $1/r^2$ potential. The work on higher Green functions was motivated by physics. In particle physics, one looks at **k-point functions** $\langle \Omega, \phi(x_1) \dots \phi(x_k), \Omega \rangle$ or Green functions to calculate the out come of particle interactions. This is usually done using a series expansion and estimated using Feynman diagrams. But in the discrete, there are no infinities, everything is finite. The 1-point function $k = 1$ case can be thought of as vacuum expectation. The 2-point functions $k = 2$ could have relations with probability amplitudes for particles to go from x_1 to x_2. The general case k could relate to scattering amplitudes of k particles. It is important to note however that all what is done here is just geometry and mathematics, actually combinatorics. Physics is motivation however.

Structures

A **finite simple graph** is a pair (V, E), where V is a finite set and $E \subset \{\{a,b\}, a, b \in V, a \neq b\}$ is a set of pairs of points. Elements in V are called **vertices**. Elements in E are called **edges**. Given a subset W of V, the **graph induced by** W is the graph $(W, \{(a,b) \in E, a \in W, b \in W\})$. For example, if W is the set of vertices that are connected directly to a vertex v, it induces the **unit sphere** $S(v)$.

A graph $\Gamma = (V, E)$ comes naturally with a simplicial complex G, where G are the vertex sets of complete subgraphs of Γ. The **maximal dimension** of Γ is the maximal dimension of G. A triangle-free graph with at least one edge has dimension 1. A graph without edges has dimension 0. The empty graph has dimension -1. A tetrahedron K_4 has dimension 3. The surface of the tetrahedron, is the 2-skeleton complex of K_4 consisting of all simplices of dimension 2 or lower. If $G = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{3,1\},\{1,2,3\}\}$ is K_3, its 2-skeleton complex is $H = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{3,1\}\}$. A complex G is called **contractible** if there exists x such that $S(x) = \overline{U}(x) \setminus U(x)$ and $G \setminus U(x)$ are both contractible with the induction assumption that $1 = \{1\}$ is contractible. The complex K_5 is a contractible complex of maximal dimension 4, the 2-skeleton complex H is not contractible.

A finite abstract simplicial complex G defines a finite simple graph $\Gamma = (V, E)$ in which $V = G$ are the vertices and a pair (a, b) is in the edge set E, if $a \subset b$ or $b \subset a$. The complex G could also be defined to be contractible if its graph is contractible. For simplicial complexes, there are more contraction steps: G is contractible, if there is a simplex $x \in G$ for which both the unit sphere $S(x) = \overline{U(x)} \setminus U(x)$ as well as $G \setminus U(x)$ are contractible simplicial complexes. If G is the Whitney complex of a graph $\Gamma = (V, E)$ and $x = v$ is a 0-dimensional simplex, then $G \setminus U(x)$ is the Whitney complex of $\Gamma \setminus v$.

If (V, E) is a graph, we can look at the graph of its complex G. This is the **Barycentric refinement** of (V, E). The Barycentric refinement of a complete graph K_{q+1} is a q-manifold with boundary. The Barycentric refinement of a triangle K_3 for example is a wheel graph with 7 vertices. If G is a complex, the Barycentric refinement is done by first looking at its graph Γ and then taking the Whitney complex of that graph. In any case, the combination of producing a graph from a complex and producing a complex from a graph is a Barycentric refinement.

Given two finite simple graphs A, B, the **join** is the graph $A \oplus B = (V, E)$, where $V = V(A) \cup V(B)$ and $E = E(A) \cup E(B) \cup \{(a, b), a \in V(A), b \in V(b)\}$ are the edges. The simplicial complex G of $A \oplus B$ is the join of the simplicial complexes of A and B. Every simplex $x \oplus y$ of G is a join of two simplices x and y. The f-vector of G encodes the cardinalities $f_k = |G_k|$ of k dimensional simplices in G. It defines the simplex generating function $f(t) = 1 + f_0 t + \cdots + f_q t^{q+1}$. We have $f_{A \oplus B}(t) = f_A(t) f_B(t)$. On the simplicial complex level, the join of two simplicial complexes G, H is the union $G \cup H \cup \{x \cup y, x \in G, y \in H\}$. If G, H are the Whitney complexes of graphs, then their join is the Whitney complex of the graph join.

A **finite set** is a collection G of finite objects. It is a standard assumption in set theory that a set does not record multiplicities of elements. The object $\{a, a, b, a, b\}$ for example is identified with $\{a, b\}$ if it is considered to be a set. If multiplicity should be needed, one could write it as a multi-set or a function $\{a, b\} \to \mathbb{N}$ encoding the multiplicities. If the order should matter, one could write it as a **finite sequence** $\{a, a, b, a, b\}$ or a **word** $aabab$ over an alphabet.

If V is a finite set, then $G = 2^V$ is the **power set** of V, the set of all subsets of V. For $V = 0$, one gets $G = 2^0 = 1 = \{0\}$, the 1-point set. This set has a **partial order structure** $A \leq B$ in the form of $A \subset B$. It is also standard in set theory not to assume the subset operation to be strict. The statement $A \subset A$ for example is always true. The set 2^V contains the empty set $\emptyset$. If $V = \{\} = \emptyset$ is the empty set, then $2^V = \{\emptyset\} = 1$ is a set with one element. Let $|A|$ denote the **cardinality** of the set. The cardinality is a non-negative integer $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. We would write $\mathbb{N}^+$ for the positive integers and then there are the integers $\mathbb{Z}$, the group completion of $\mathbb{N}$. We have $|0| = 0, |1| = 1$ and $|2^A| = 2^{|A|}$, justifying the notation. There is a natural identification of 2^{A+B} with the **Cartesian product** $2^A \times 2^B$ because if x is a subset of $A + B$, then x defines (a, b) with $a = x \cap A$ and $b = x \cap B$. But $(a, b) \in 2^A \times 2^B$.

There is a monoid operation on the category of finite sets. It is the **disjoint union** $+$, and is also known as **co-product**. It is associative and so a **monoid structure**. The empty set $\emptyset$ is the zero element. If this monoid is group completed to become a group, one has then not only sets but "negative sets". If the co-product is **group completed** with negative sets, one has an additive group.

The **Cartesian product** $A \times B$ is defined as the set of all ordered pairs $\{(a, b), a \in A, b \in B\}$. The singleton set 1 is the 1-element in this multiplication. The **Cartesian ring** $(FinSet, +, 0, \times, 1)$ is a commutative ring with 1 if one identifies $(-A) \times B = A \times (-B) = -(A \times B)$. The category FinSet so naturally extends to a commutative ring in which 1 is the one-element and where 0 is the zero-element. If the cardinality function is extended to negative sets as $|-A| = -|A|$, the cardinality function defines a homomorphism from the ring of finite sets to $\mathbb{Z}$. If sets of the same cardinality are identified, then this is an isomorphism.

A **finite abstract simplicial complex** G is a finite set of non-empty sets that is closed under the operation of taking non-empty subsets. This geometric structure has only one axiom, the **hereditary axiom**. This simplicity makes it appealing. There is a prise to pay however. Sets of sets are less intuitive than graphs.

Given a simplicial complex, one can assume that every $x \in G$ comes with an order. The choice of the order corresponds to picking a coordinate system and is irrelevant. It does not matter what order we chose for each x, but it can be important to have an order in order to do linear algebra. We do not insist that the order is inherited by subsets. There is no order on maximal simplices in general which works like that, the simplest example being the Moebius strip complex, which is the simplicial complex of the graph complement of C_7. If G contains subsets of integers, we can take for example the natural order. We also assume that the elements of G are ordered according to dimension to get block matrices. We for example write $G = \{\{1\}, \{2\}, \{1, 2\}\}$ the complex of K_2 and not $G = \{\{1,2\},\{1\},\{2\}$. Again, this is just a choice of labeling a basis when doing linear algebra and irrelevant for any geometry.

As custom in set theory, sets contain different elements. $G = \{\{1,2\},\{1\},\{2\}\}$ is an example of a simplicial complex. The structure $G = \{\{1\}.,\{1,1\}\}$ would be identified with $\{\{1\},\{1\}\}$ and so with $\{\{1\}\}$. A set does not record multiplicities. If this is needed, like for multi-graphs, we would use sequences. For example, the sequence $(\{1\},\{2\},\{1,2\},\{1,2\})$ is a quiver with two edges connecting 1 with 2.

An equivalent definition of simplicial complexes that is often seen in the literature invokes the **vertex set** $V = \bigcup_{x \in G} x$. The assumption is then that G is a subset of $2^V \setminus \{\{\}\}$ that does not contain the empty set and that satisfies the heriditary axiom. In that frame work, a simplicial complex is a pair (V,G) where $G \subset 2^V$.

More general than a simplicial complex is a **set of sets**. This structure has also been called a **hypergraph**. A hypergraph in general does not allow a reasonable calculus in the sense that one can not define a non-trivial exterior derivative. A hypergraph also is allowed to contain the empty set. In order for the theory to work properly, it is important not to let the empty set to be in. The empty set can however be a structure itself. It is the zero element in all structures. The heriditary assumption is needed to get a notion of derivative that works.

There are flavors of combinatorial topology that include $\emptyset$ into G. The **Euler characteristic** $\chi(G) = \sum_{x \in G} \omega(x)$ with $\omega(x) = (-1)^{\dim(x)}$ for example would not work. The genus $1 - \chi(G)$ is popular as it renders the join multiplicative.

There are compelling reasons to keep $\chi(G)$ the Euler characteristic as we want compatibility with addition (disjoint union) and multiplication (Cartesian product). There are also reasons to assume that simplices are contractible objects in the sense that its closure $\overline{\{x\}}$ is contracdtible. One should distinguish the element $x \in G$ with $K = \overline{\{x\}} \subset G$ even so they can be naturally identified. The object K is simplicial complex of Euler characteristic 1. And then there is $U = \{x\}$ which is an open set in K. The complement $K \setminus U$ is the **boundary complex** of K. It is closed, a simplicial complex and always a sphere. Spheres are always non-contractible objects of Euler characteristic $1 + (-1)^q$. The void 0 is the (-1) sphere. It is the boundary complex of the 1-point complex $1 = \{1\}$. It has Euler characteristic 0 and is not contractible.

Simplicial complexes can be defined from other structures. Any graph defines a **Whitney complex**, in which the vertex sets of complete subgraphs are the sets. It is also called the **flag complex** or **order complex**. The empty graph generates the void. It is not a complete complex as a simplicial complex never contains the empty set. An other complex is the **forest complex**, in which the edges of a forest are the sets. A forest decomposes into trees and trees are assumed in this set-up to be one-dimensional, meaning to have maximal dimension 1. Single points = seeds, are not yet considered trees. In other contexts, like for the matrix forest theorem, it is helpful to include seeds as trees.

Simplicial complexes can be derived from already given simplicial complexes. The intersection of two simplicial complexes is a simplicial complex. If two simplicial complexes are disjoint, then their intersection is the void, which is a simplicial complex. The union of two simplicial complexes $G = A \cup B$ is a simplicial complex as checking the definition shows. If $x \in G$, then either $x \in A$ or $x \in B$ so that the hereditary axiom holds. If G is a simplicial complex of dimension q, the **skeleton complex** of dimension $k \leq q$ is the set of sets for which the dimension is smaller or equal than k.

Given an arbitrary set A of non-empty sets, one can look at the **closure** $\overline{A}$ of A. It is the smallest simplicial complex containing A. To get $\overline{A}$, just adjoin all the non-empty subsets of A. The name closure is adequate because it is the closure operation in the Alexandroff topology, the topology first considered by Pavel Alexandrov.

A **topological space** $(X, \mathcal{O})$ is a set X with a collection $\mathcal{O}$ of subsets of X with the property that both $\emptyset$ and X are in $\mathcal{O}$ and that $\mathcal{O}$ is closed under finite intersections and closed under arbitrary unions. A topological space is **finite**, if X is a finite set. A **base** $\mathcal{B}$ of a topological space is a subset of $\mathcal{O}$ that generates $\mathcal{O}$ in the sense that every $A \in \mathcal{O}$ can be written as a union of base elements, where the empty union $\bigcup_{A \in \emptyset} A = \emptyset$. The base does not have to contain the empty set $\emptyset$.

The **Alexandroff topology** is the topology generated by the base $\{U(x)\}_{x \in G}$, given by the **stars** $U(x) = \{y \in G, x \subset y\}$. A **sub-base** is $\{U(v)\}_{x \in V = G_0}$ because intersections of the sub-base produce base elements. If G has positive maximal dimension, the topology is non-Hausdorff. Given an edge (a, b), the points a, b can not be separated by open sets as every open $U(a) \cap U(b) = U((a, b))$ is open and not-empty.

An arbitrary topology has the **Alexandroff property**, if arbitrary intersections of open sets are open. Every finite topology automatically has the Alexandroff property. If the abstract simplicial complex is not finite but has a finite maximal dimension, then still, its topology has the Alexandroff topology.

A topology $(G, \mathcal{O})$ is **Kolmogorov** (T0) if for any two distinct points $x, y \in G$, there exist neighborhoods $U(x), U(y)$ such that either $x \notin U(y)$ or $y \notin U(x)$. A topology is **Fréchet** (T1) if for any two distinct points $x, y \in G$, there exist neighborhoods $U(x), U(y)$ such that $x \notin U(y)$ and $y \notin U(x)$. A topology is **Hausdorff** (T2) if for any two distinct points $x, y \in G$, there exist neighborhoods $U(x), U(y)$ such that $U(x) \cap U(y) = \emptyset$.

The topology on a simplicial complex G is always **Kolmogorov**. Proof: given $x \neq y \in G$. If there is an inclusion like $x \subset y$, then $U(y) \subset U(x)$ but $x \notin U(y)$. If there is no inclusion, then both $x \notin U(y)$ and $y \notin U(x)$. But a topology on G is neither **Fréchet**, nor **Hausdorff** if its maximal dimension is positive. If $x \subset y$, then y is always in $U(x)$. As any finite topology, it is **Alexandroff**, meaning that there are smallest neighborhoods $U(x) = \{y \in g, x \subset y\}$ of every $x \in G$. A finite topological space that is Hausdorff is necessarily the discrete topology, in which all subsets are open. Finite topologies are never Hausdorff unless they are zero-dimensional.

The Alexandroff topology of a simplicial complex is close to the **Zariski topology** because the closed sets are the sub-simplicial complexes. We could associate sub-simplicial complexes of maximal dimension q which have the property that all unit spheres have maximal dimension $q-1$ as some sort of variety.

Define the **unit ball** $B(x) = \overline{U(x)}$ and the **unit sphere** $S(x) = B(x) \setminus U(x)$. The later is the boundary of the unit ball and is again closed as the intersection of a closed set $B(x)$ with a closed set $U(x)^c$. If $x = v$ is 0-dimensional, the unit sphere $S(x)$ is sometimes called the **link**.

Note that G is a set of sets and that elements in $\mathcal{O}$ are a set of subsets of G. The open sets in $G = \{\{1\},\{2\},\{1,2\}\}$ are $\{\emptyset, \{\{1\},\{1,2\}\}, \{\{2\},\{1,2\}\}, \{\{1,2\}\}, G\}$. So, it is important to distinguish a simplex $x \in G$ with $\{x\}$ which is in general neither open nor closed. Maximal simplices x produce the open set $U(x) = \{x\}$, vertices x produce closed sets $C(x) = \{x\}$. If x is an intermediate dimension, then $\{x\}$ is neither open nor closed. The smallest closed set containing x is $\overline{\{x\}}$, the smallest open set containing x is $U(x)$.

One could also look at topologies on the vertex set V but that would be different. For example, if G is the complex of K_4 with vertex set $V = \{1,2,3,4\}$, then the pseudo circle $\mathcal{O} = \{(1,2,3,4),(1,2,3),(1,2,4),(1,2),(1),(2),(0)\}$ a subset of G, not a set of sets from G. This example is known as the **small circle**. It is neither an open set nor a closed set in G.

If G is a finite abstract simplicial complex, a **contraction step** is defined as a removal of a star $U(x)$ with contractible boundary $S(x) = \delta U(x)$ from G. Formally, this means $G \to G \setminus U(x)$ under the assumption that $S(x) = \overline{U(x)} \setminus U(x)$ is contractible. The inverse operation is called an **expansion step**. Two complexes are called **homotopic** if there exists a finite number of homotopy expansion or contraction steps, deforming one to an other.

If G can be reduced to 1 by a finite number of contraction steps, the complex G is called **contractible**. This is very different from being homotopic to 1. The **dunce hat** is an example of a complex that is homotopic to 1 but not contractible.

If G is contractible and H is arbitrary, then $G \oplus H$ is contractible. For example $G \oplus 1$, the cone extension is always contractible. Every ball $B(x) = \overline{U(x)}$ is contractible. Contractible simplicial complexes form a **semi-group** within the monoid of all simplicial complexes. It is not a sub-monoid because the zero element is not considered contractible.

No q-manifold is contractible unless it is the zero-dimensional complex $G = 1$, the one point complex. No sphere is contractible: the empty set which is the (-1)-sphere is not contractible by definition. Recursively, no q-sphere for $q > 0$ is contractible. The semi-group of contractible complexes is disjoint from the sphere monoid.

Contractibility can be computed in a time complexity that grows maximally quadratically in the number of elements of G: just go through all the $x \in G$ and check in each case whether $S(x)$ and $G \setminus U(x)$ are both contractible. Both simplicial complexes $S(x)$ and $G \setminus U(x)$ are smaller.

On the other hand, there is no fixed Turing machine that takes as an input a simplicial complex G and tells whether G is homotopic to 1 or not. The distinction between contractibility and **homotopic to 1** could not be bigger. This is a reason not to use words like **collapsibility** and **contractibility** to make the distinction since both terms are in colloquial language associated with a shrinking process. But homotopic to 1 can mean that we have to take a complex and blow it up again and again until it eventually can be shrunk to a point.

Fix a finite abstract simplicial complex G. A linear functional X on the set of f-vectors $\phi(G) = X \cdot f(G)$ is called a **valuation** on G. It satisfies $\phi(A \cup B) + \phi(A \cap B) = \phi(A) + \phi(B)$ for closed subsets.

The Hadwiger theorem statement that the set of valuations on a complex of maximal dimension q has dimension $q+1$ follows from the definition. A basis is given by the counting valuations $f_k(G)$. An example of a valuation is the Euler characteristic $\chi(G) = f_0 - f_1 + f_2 \cdots (-1)^q f_q$.

A valuation X that satisfies $X(A) = 1$ for every complete sub-complex A of G is $\chi(G)$. Proof: If G has maximal dimension q, then the space of valuations has dimension $q+1$. Valuations are of the form $X(A) = X \cdot f(A)$. If $X(A) = 1$ for every complete graph, then $X \cdot f(K_k) = 1$, for the f-vectors $f(K_k)$ of K_k. But since these vectors form a basis in the vector space $\mathbb{R}^{d+1}$, we have uniqueness and $X = \chi$.

Let (G, G) be the set of all pairs of simplices (x, y) with $x, y \in G$ such that $x \cap y$ is non-empty. More generally, let (A, B) be the set of all pairs of simplices (x, y) with $x \cap y \neq 0$ and $x \in A, y \in B$. The **f-matrix** of (A, B) is $f_{k,l}(A, B)$, the number of elements (x, y) in (A, B) with x of dimension k and y of dimension l.

The f-matrix defines a multi-linear valuation. The vector space of quadratic valuations has dimension $(q+1)(q+2)/2$ as the numbers $f_{k,l}$ are symmetric, this is the dimension of the space of symmetric $(q+1) \times (q+1)$ matrices.

Valuations generalize counting or measure with some sort of symmetry. In the continuum, one restricts the theory of valuations to finite unions of convex sets in $\mathbb{R}^n$. The theory of measures does not look at the internal structure of the sets which are measured, unlike the theory of valuations which look inside. In the continuum, one needs to look at the internal k-dimensional structure of a set using probabilistic tomography, like the Crofton formula. One can for example look at the length of a convex set in $\mathbb{R}^2$ as an expectation of length of intersections with random lines. This satisfies the valuation property because for every experiment, we have the valuation property.

A reasonable **geometry on a finite set** need a **category** that enables basic operations from calculus. We especially like to build products, quotients or level sets. Any of these operations extends the scope of simplicial complexes in general making it necessary to work in a larger category.

The category *Set* of finite sets is an **elementary topos**: cartesian product,inclusion, exponentials and subobject classification works. There is a natural generalization of this in the form of **delta sets**.

A finite category C defines the pre-sheaf category $Pre(C) = Set^{C^{op}}$. **Delta sets** are the pre-sheaf category of the simplex category C with strict inclusions as morphisms.

A delta set is a sequence of sets $G_0, G_1, \dots, G_q$ with **face maps** $d_i^k : G_{k+1} \to G_k, i = 0, k+1$ that satisfy $d_i^k d_j^{k+1} = d_{j-1}^k d_i^{k+1}$ for $i < j$. A simplicial complex is a delta set The simplest example of a delta set that is not a simplicial complex is $G_0 = \{0\}, G_1 = \{(0,0)\}$ with maps $d_0((0,0)) = 0, d_1((0,0)) = 0$. Open sets in a simplicial complex are delta sets, level sets of functions are delta sets. Products of simplicial complexes are delta sets.

An **abstract delta set** is a finite set G of n sets with a dimension function $R : G \to \{0, \dots, q\}$ and a matrix $D = d + d^*$, where df is supported on $G_{k+1} = \{R = k+1\}$ if f is supported on G_k. Any delta set defines an abstract delta set. Abstract delta sets are sets of non-empty sets with a notion of dimension and derivative. While one can avoid delta sets by going to a refinement and look at the Whitney complex of the graph defined by the delta set, there are computational advantages to leave the delta set structure.

The Dirac operator D of a Cartesian product for example is much smaller than the Dirac operator of the Barycentric refined case. Similarly, the cohomology of a level set in a simplicial complex can be determined faster than using the Whitney complex of the graph of this structure. A third example is looking at quotients G/A where A is a subgroup of the automorphism group of a simplicial complex G. The quotient is often no more a simplicial complex.

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