ELEMENTS OF FINITE GEOMETRY

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Unit 12: Higher Green

The k-point **Green function** $g_m(x_1,\dots,x_k) = \prod_{j=1}^k \omega(x_j)\chi_m(\bigcap_{j=1}^k U(x_j))$ for the m**'th characteristic** $\chi_m(A) = \sum_{x\in A^m, \bigcap_j x_j \in A} \prod_{j=1}^m \omega(x_j)$ of $A \subset G$ satisfies $\chi_m(G) = \sum_{x\in G^k} g_m(x_1,\dots,x_k)$.

12.1. The m**'th characteristic** is defined as $\chi_m(G) = \sum_{\mathbf{x}\in G^m, \bigcap \mathbf{x}\in G} \omega(x)$. It is Euler characteristic in the linear case $m = 1$ and quadratic characteristic in the quadratic case $m = 2$. If used vector notation $\mathbf{x} = (x_1,\dots,x_m)$ and multi-index notation $\omega(\mathbf{x}) = \prod_{j=1}^m \omega(x_j)$ and $\bigcap \mathbf{x} = x_1 \cap x_2 \cap \cdots \cap x_m$.

12.2. The k**-point Green function** on G^k is $g_m(\mathbf{x}) = \omega(\mathbf{x})\chi_m(\bigcap U(\mathbf{x}))$, where $U(x_j)$ is the star of x_j and the star $U(\mathbf{x})$ of $\mathbf{x}\in G^k$ is the intersection of the stars of $x_1,\dots,x_k$. The k-tuple $\mathbf{x} = (x_1,\dots,x_k)$ does not need to intersect but the k-point Green function are still local because $U(\mathbf{x})$ is empty, if one of the "particles" x_j is out of reach.

12.3. The Green function identity $\chi(G) = \sum_{x,y} g(x,y)$ for the quadratic case $m = 2$ generalizes but already in the cubic case $g(x,y,z)$ is a tensor. For $m = 2$, we could think of $g(x,y)$ as a matrix, where it was the inverse of $L(x,y)$. While $\chi_m(G)$ involves only m-tuples of G that intersect, the Green functions $g(x_1,\dots,x_k)$ involve all k-tuples of G. Think of a k**-point potential energy**. The energy theorem generalizes to

Theorem: $\chi_m(G) = \sum_{\mathbf{x}\in G^k} g_m(\mathbf{x})$.

Proof. Generalize to energized complexes where $\omega(\mathbf{x})$ is replaced with a general function $h_m(x_1,\dots,x_m)$ of m variables producing a m-linear valuation. Since $h_m \to g_k$ is linear, we only need to verify the statement in the simplest possible case, where h is 1 only for a single configuration $\mathbf{z} = (z_1,\dots,z_m)$ and 0 else. The left hand side is then 1. On the right hand side, we have to look at all $\mathbf{x} = (x_1,\dots,x_k)$ for which $\mathbf{z} \in U(\mathbf{x}) = U(x_1) \cap U(x_2) \cap \cdots \cap U(x_m)$. $\sum_{\mathbf{x}\in G^k} \omega(\mathbf{x})\chi_m(U(\mathbf{x})) = \prod_{j=1}^k \sum_{x_j\in G} \omega(x_j)\chi_m(U(x_j)) = 1$. $\square$

12.4. A second major point is the **sphere formula** for $S(\mathbf{x}) = \delta U(\mathbf{x}) = B(\mathbf{x}) \setminus \mathbf{U}(\mathbf{x})$, where we write similarly as $U(\mathbf{x}) = \bigcap_j U(x_j)$ also $B(\mathbf{x}) = \bigcap_j B(x_j)$.

Theorem: $\sum_{\mathbf{x} \in G^k} \omega(\mathbf{x}) \chi_m(S(\mathbf{x})) = 0$.

12.5. This means that the total m'th characteristic of all unit spheres of even k-point configurations is the same than the total m'th characteristic of all unit spheres of odd k-point configurations.

Proof. The energy theorem also works of $U(\mathbf{x})$ is replaced by $B(\mathbf{x})$, which is the closure of $U(\mathbf{x})$. The two equations $0 = \sum_{\mathbf{x} \in G^k} \omega(\mathbf{x}) \chi_m(U(\mathbf{x}))$ and $0 = \sum_{\mathbf{x} \in G^k} \omega(\mathbf{x}) \chi_m(B(\mathbf{x}))$ and the local valuation formula blow prove the theorem. $\square$

12.6. For characteristic $m = 1$, the formula $\chi(\overline{U}) = \chi(U) + \chi(\delta U)$ follows from the fact that every simplex in $\overline{U}$ is either in U or δU. The local valuation formula is:

Theorem: $\chi_m(B(\mathbf{x})) = \chi_m(U(\mathbf{x})) - (-1)^m \chi_m(S(\mathbf{x}))$.

12.7. For odd m this means $\chi(U) + \chi(\delta U) = \chi(\overline{U})$. For quadratic characteristic $m = 2$, one has $\chi_2(U(x)) - \chi_2(S(x)) = \chi_2(B(x))$. If $B(x)$ is a replaced with a manifold G with boundary and we chose $k = 1$, one gets the boundary formula $\chi_2(G) = \chi_2(\text{int}(G)) - \chi_2(\delta G)$ which then simplifies to $\chi_2(G) = \chi(G) - \chi(\delta G)$.

Theorem: If G is manifold we have $\chi_m(G) = \chi(G) - (-1)^m \chi(\delta G)$.

Proof. Use induction and that the left hand side is $\chi_m(\text{int}(G)) - (-1)^m \chi_m(S(\delta G))$. $\square$

12.8. So, also the higher characteristics of manifolds with boundary can be expressed using Euler characteristic. For even-dimensional manifolds with boundary $\chi_m(G) = \chi(G)$. For odd dimensional manifolds with boundary $\chi_m(G) = -(-1)^m \chi(\delta G)/2$. For even dimensional balls G for example, where the boundary is an odd-dimensional sphere, $\chi_m(G) = 1$ independent of m, while for odd dimensional balls $\chi_m(G) = (-1)^{m+1}$.

12.9. Higher characteristic $\chi_m(G)$ can be written as $\chi_m(G, G, \dots, G)$ if one defines the multi-linear expressions $\chi_m(A_1, \dots, A_m) = \sum_{x_1 \cap \cdots \cap x_m \in G, x_i \in A_i} \omega(\mathbf{x})$. Like for multi-linear forms in linear algebra, fixing all but one coordinates produces a valuation. Like Euler characteristic had the f-vector and quadratic valuations had the f-matrix f_{kl}, multi-linear forms define the **f-tensor** $f_{k_1, \dots, k_m}$. Applied to m Dehn-Sommerville vectors, we get Dehn-Sommerville invariants for manifolds answering a question of Gruenbaum. The most prominent Dehn-Sommerville invariant is $X = (1, -1, 1, -1, \dots)$ which produces characteristic invariants $f(X, X, \dots, X)$. Let us abbreviate m-valuation for multi-linear valuation of m variables:

Theorem: χ_m is the only m-valuation that is Barycentric invariant.

Proof. $(1, -1, 1, -1, \dots, \pm 1)$ is the only eigenvector of the **Barycentric operator**. $\square$

Quantities with this property are **topological invariants**. The characteristic invariants are not homotopy invariant for $m \geq 2$. Balls of odd dimensions have a different invariant than balls in odd dimensions. But all balls are contractible and especially homotopic to 1.

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