ELEMENTS OF FINITE GEOMETRY

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Unit 11: Quadratic cohomology

If G is a finite abstract simplicial complex, the quadratic characteristic $\chi_2(G) = \sum_{x \cap y \in G} \omega(x)\omega(y)$ is a quadratic version of Euler characteristic $\chi(G) = \sum_x \omega(x)$. It comes with a cohomology similarly as $\chi(G)$ comes with simplicial cohomology. For q-manifolds with boundary $\chi_2(G) = \chi(G) - \chi(\delta G)$.

11.1. Classical theorems like Gauss-Bonnet, Poincare-Hopf, Euler-Poincare or Brouwer-Lefschetz are related to Euler characteristic $\chi(G) = \sum_{x \in G} \omega(x)$, both in the continuum as in the discrete. There are higher order theories for which **quadratic characteristic** $\chi_2(G) = \sum_{x \cap y \in G} \omega(x)\omega(y)$ is the simplest. Some theorems generalize directly but there are changes, the most important being that χ_2 is of a topological nature. It is not homotopy invariant. If G is the closure of $\{x\}$ then $\chi_2(G) = \omega(x) = (-1)^q$. For manifolds without boundary, $\chi_2(G) = \chi(G)$.

11.2. Quadratic cohomology works with functions on **pairs of intersecting simplices** $(x, y) \in G \times G$. The **exterior derivative** is defined as $df(x, y) = d_x f(x, y) + \omega(x) d_y(f, y)$, where d_x, d_y are the partial simplicial exterior derivatives. Cubic, quartic and higher characteristics and cohomology are defined similarly.

11.3. With $\dim(x, y) = \dim(x) + \dim(y)$, let $\Lambda_k = \{f : G_k \to \mathbb{R}$, where $G_k = \{(x, y), \dim(x, y) = k\}$ is the space of k-forms and let f_k denote its dimension. Then $w(G) = \sum_{k=0}^{2q} (-1)^k f_k$. The space $\Lambda = \oplus_{k=0}^{2q} \Lambda_k$ of **differential forms** has as dimension the number of ordered pairs $(x, y) \in G^2$ with $x \cap y \in G$. It is a finite dimensional vector space and $\mathrm{str}(1) = \chi_2(G)$ is the quadratic characteristic. The cardinalities of the quadratic complex is encoded by $f_{kl} = \{(x, y), x \cap y \neq \emptyset, \dim(x) = k, \dim(y) = l\}$.

11.4. The Dirac operator $D = d + d^*$ and the Hodge operator $H = D^2 = dd^* + d^*d$ are defined as before. If G has maximal dimension q, then H has $2q + 1$ blocks H_k. It belongs to forms $f(x, y)$ acting on pairs (x, y) of simplices with $\dim(x) + \dim(y) = k$. The kernel of H_k is called the k'th Betti number b_k. As in the linear case, we can split $\Lambda(G)$ into **even forms** Λ_{2j} and **odd forms** Λ_{2j+1}. The matrix D is an isomorphism between even and odd forms on the image of H so that $\mathrm{str}(H^n) = 0$ for $n > 0$. The analog of Euler-Poincaré is proven in exactly the same way using heat deformation.

Theorem: $\chi_2(G) = \sum_{k=0}^{2q} (-1)^k f_k = \sum_{k=0}^{2q} (-1)^k b_k$.

11.5. An automorphism T of G also induces an automorphism on the intersection diagonal of $G \times G$ by $T(x,y) = (T(x), T(y))$ the reason being that if $x \cap y \neq \emptyset$, then also $T(x) \cap T(y) = T(x \cap y) \neq \emptyset$. The Lefschetz number $\chi_{2,T}(G)$ is as in the linear case the super trace of the Koopman operator U on cohomology. The index of a fixed point (x,y) is $i_T(x,y) = \omega(x)\omega(y)\text{sign}(T|x)\text{sign}(T|y)$. Also the Lefschetz fixed point theorem generalizes with the same heat proof to a **quadratic Lefschetz fixed point theorem**:

Theorem: $\chi_{2,T}(G) = \sum_{(x,y) \in F} i_T(x,y)$.

11.6. It generalizes the Euler-Poincaré theorem because for $T = Id$, $\chi_{2,T}(G) = \chi_2(G)$ and every (x,y) with $x \cap y \neq \emptyset$ is then a fixed point of T with index $\omega(x)\omega(y)$ so that the right hand side in the Lefschetz theorem is the definition of $\chi_2(G)$.

11.7. Also Gauss-Bonnet, Poincaré-Hopf and index expectation generalize. Curvature $K(v)$ for quadratic valuation is $K(v) = \sum_{x \sim y} \omega(x)\omega(y)/|x|$.

Theorem: $\chi_2(G) = \sum_{v \in V} K(v)$.

11.8. For manifolds with boundary, we know how to compute it.

Theorem: $\chi_2(G) = \chi(G) - \chi(\delta G)$.

For even q, this gives $\chi_2(G) = \chi(G)$ even with boundary. For odd q, this gives $\chi_2(G) = -\chi(\delta G)/2$.

11.9. Examples:
1) For $P_2 = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\}\}$, there 15 interacting pairs (x,y) where $f_0 = 3, f_1 = 8, f_2 = 4$. The Wu characteristic is $3 - 8 + 4 = -1$. 2) The utility graph $G = K_{3,3}$ has Euler characteristic $\chi(G) = -3$ and Wu characteristic $\chi_2(G) = 15$. It is maximal among all graphs with 6 vertices.
3) For a star graph with n rays, the Wu characteristic is $n^2 - 3n + 1$. For example, for $n = 0$, it is 1, for $n = 1$ it is -1 for $n = 9$ it is 55. All star graphs are contractible illustrating the non-homotopy invariance.
4) Adding a hair to a 2 sphere reduces the Wu characteristic by 2.
5) The Wu characteristic of the cube graph is 20, the Wu characteristic of the dodecahedron is 50. Both graphs have no triangles and constant vertex degree $d = 3$ so that in both cases, the curvature is constant $5/2$.
6) The dunce hat has quadratic cohomology $b = (0,0,0,1,2)$.

7) The Moebius strip has $b = (0, 0, 0, 0, 0)$ while the cylinder has $b = (0, 0, 1, 1, 0)$. The cohomology sees the twist. The Lefschetz fixed point theorem implies that if an automorphism preserves harmonic 2-forms and reverses harmonic 3-forms, then there are at least 2 fixed points. We once speculated that the preservation of harmonic 2-forms could be related to area preservation and the twisting of harmonic 3 forms be related to twist so that quadratic cohomology could give Birkhoff's fixed point theorem.

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