ELEMENTS OF FINITE GEOMETRY

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Unit 10: Index formula

If g is a vertex coloring of a graph (V, E), the level set $S_g(v) \subset S(v)$ produces the **symmetric Poincare-Hopf index** $j_g(v) = 1-\chi(S(v))/2-\chi(S_g(v))/2$. This implies that odd-dimensional manifolds are flat and that for 4-manifolds, curvature $K(v)$ is the genus expectation of a random 2-manifold in the 3-manifold $S(v)$.

10.1. A finite abstract simplicial complex G, it defines a graph (V, E) a Poincaré-Hopf result for G can be formulated as the Poincaré-Hopf theorem for the corresponding graph with $V = G$. A locally injective function $g : G \to R$ is a vertex coloring function $g : V \to R$. The index $i_g(v) = 1 - \chi(S_g^-(v))$ defines then the **symmetric index** $j_g(v) = [i_g(v)+i_{-g}(v)]/2$. Since $\text{sign}(g(w)-g(v))$ is a $\{-1, 1\}$-valued function on $S(v)$, we can look at its level set $S_g(v)$ in $S(v)$. If $S(v)$ was a $(q-1)$-sphere, then $S_g(v)$ is a $(q-2)$-manifold.

10.2. The following result combines the Poincaré-Hopf with the discrete Sard story showing that for manifolds, the symmetric index is expressible as the Euler characteristic of submanifolds. First in general:

Theorem: If g is a coloring, then $j_g(v) = 1 - \chi(S(v))/2 - \chi(S_g(v))/2$.

Proof. The level set result clarifies what we mean with $S_g(v)$: it is a level set in $S(v)$. We have a decomposition $S(v) = S_g^-(v) \cup S_g^+(v) \cup S_g(v)$ in the sense that every simplex in $S(v)$ is either a simplex in $S_g^-(v)$ or a simplex in $S_g^+(v)$ or a simplex in $S_g(v)$. Therefore, $\chi(S(v)) = \chi(S_g^-(v)) + \chi(S_g^+(v)) - \chi(S_g(v))$, the negative sign coming from the fact that $S_g(v)$ has edges as smallest dimensional simplices, shifting dimensions by 1. Therefore, $\chi(S(v)) + \chi(S_g(v)) = 1 - i_g(v) + 1 - i_{-g}(v) = 2 - 2j_g(v)$. Now solve for $j_g(v)$. $\square$

10.3. In the case of manifolds, we have by the Euler gem formula for odd q that $j_g(v) = -\chi(S_g(v))/2$ which means that $j_g(v) = 0$. Also the sphere formula had shown that the odd-dimensional manifold $S_g(v)$ had zero Euler characteristic. Therefore, for any index expectation curvature that comes from a probability measure, for which $g \to -g$ is an isometry.

Theorem: Odd-dimensional manifolds are flat.

10.4. In the case of even dimensional manifolds we have $j_g(v) = 1 - \chi(S_g(v)))/2$. Motivated from notions in 2-dimensions, let us call $g = 1 - \chi(G)$ the **genus** of G. It is also known as a **reduced Euler characteristic**.

Theorem: Curvature $K(v)$ in even dimensions is the genus average of $q-2$ manifolds in $S(v)$.

10.5. For a 4-manifold for example, where $S_g(v)$ is an orientable manifold of genus g with Euler characteristic $\chi = 2 - 2g$, we have

Theorem: For a 4-manifold, the curvature at v is the expected genus of a random surface in $S(v)$.

10.6. The theorem generalizes from manifolds **Dehn-Sommerville q-manifolds**. These are q-dimensional finite abstract complexes for which all $S(x)$ are Dehn-Sommerville $(q-1)$-manifolds of Euler characteristic $1-(-1)^q$. **Euler's gem** implies that q-manifolds are q-Dehn-Sommerville manifolds. A **Dehn-Sommerville q-sphere** is a Dehn-Sommerville q-manifold of Euler characteristic $1+(-1)^q$. The sphere formula $\sum_{x \in G} \omega(x)\chi(S(x)) = 0$ implies that every odd-dimensional Dehn-Sommerville manifold is a Dehn-Sommerville sphere.

Theorem: A level co-dimension k level set defined by a function $g : V(G) \to \{0, \dots, k\}$ on a Dehn-Sommerville q-manifold G is a $(q-k)$ Dehn-Sommerville manifold or empty.

10.7. It implies :

Theorem: For odd-dimensional Dehn-Sommerville complexes, $j_g(v) = 0$ for all v. For even dimensional Dehn-Sommerville complexex $j_g(v) = (1 - \chi(S_g(x))/2$.

10.8. Let $(\Omega, \mathcal{A}, P)$ be a probability measure on locally injective functions with the property that $Tg = -g$ is an automorphism of the probability space. We call the index expectation curvature $k(v) = \mathrm{E}[j_g(v)]$ a **symmetric curvature**. The Levitt curvature and the color curvature are both symmetric. We have in particular:

Theorem: Odd-dimensional Dehn-Sommerville manifolds are flat for symmetric curvatures

10.9. Especially all odd dimensional q-manifolds are flat. This might explain why a classical Gauss-Bonnet curvature has never been formulated in odd dimensions.

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