ELEMENTS OF FINITE GEOMETRY

OLIVER KNILL

Unit 5: Euler-Poincaré

A simplicial complex defines an exterior derivative d on the Hilbert space of forms. The kernels of $H = (d + d^*)^2$ restricted to k-forms is the k'th Betti number. The alternating sum of the b_k is the Euler characteristic.

5.1. Let G be a finite abstract simplicial complex. It can be the Whitney complex of a graph for example. If G has cardinality n, the vectors in the Hilbert space $\Lambda = l^2(G) \sim \mathbb{R}^n$ are **discrete differential forms** or simply **forms**. Functions on $G_k = \{x \in G, \dim(x) = k\}$ are called k**-forms**. There is a natural decomposition $\Lambda = \oplus_{k=0}^q \Lambda_k$ and linear maps on Λ that preserve Λ_k are block matrices if we assume the elements in G to be ordered according to dimension.

5.2. For $y \subset x$ with $\dim(y) = \dim(x) - 1$, define $d(x, y) = \text{sign}(x, y)$, where $\text{sign}(x, y)$ is the sign of the permutation that maps (vy) to x, where $v = x \setminus y$. For example, $\text{sign}((1, 2, 3, 4), (1, 2, 4)) = \text{sign}((1, 2, 3, 4), (3, 1, 2, 4)) = 1$. The matrix d is nilpotent $d^2 = 0$ and is called the **exterior derivative**. It maps k-forms to $(k + 1)$-forms. The symmetric matrix $D = d + d^*$ is the **Dirac operator**. Its square $H = D^2 = dd^* + d^*d$ is the **Hodge operator**.

5.3. H decomposes into blocks $\oplus_{k=0}^q H_k$ because both dd^* and d^*d map Λ_k into itself. The k**'th cohomology group** are defined as the space of k**-harmonic forms** $\ker(H_k)$. Its dimension is the k'th **Betti number** b_k of G. If I_k are the identity matrices on $l^2(G_k)$, then $f_k = \text{tr}(I_k) = |G_k|$ is the number of k-simplices and $\chi(G) = \sum_{k=0}^q (-1)^k f_k$. Similarly as the f-vector $(f_0, f_1, \dots, f_q)$ encodes cardinalities, the Betti-vector $(b_0, b_1 \dots, b_q)$ encodes dimensions of kernels.

5.4. The **McKean-Singer symmetry** is a spectral symmetry telling that non-zero eigenvalues on even forms correspond bijectively to non-zero eigenvalues on odd forms. It can be encoded as:

Theorem: $\text{str}(e^{-tH}) = \chi(G)$.

Proof. If u is an eigenvector to a non-zero eigenvalue $Hu = \lambda u$, then $v = Du$ is an eigenvector to the same eigenvalue $Hv = \lambda v$. But if u is an even dimensional form, then v is an odd dimensional form. We have now an isomorphism between even and odd parts of the image of H. It follows that the super trace of H^k is zero for $k > 0$. As $\text{str}(H^0) = \text{str}(1) = \chi(G)$, when we apply the super trace to $e^{-tH} = \sum_{k=0}^{\infty} t^k H^k/k!$ this number is independent of t and equal to the super trace of H^0 which is $\chi(G)$. □

5.5. The following result is the **Poincaré-Hopf theorem**.

Theorem: $\chi(G) = \sum_{k=0}^{q} (-1)^k f_k = \sum_{k=0}^{q} (-1)^k b_k$.

Proof. Use the heat flow e^{-tH} and the McKean-Singer symmetry. For $t = 0$, the super trace of e^{-tH} is the left hand side. For $t = \infty$, we get the right hand side. Since the super trace does not change, the left and right hand side agree. □

5.6. The **Hodge decomposition** tells that any k-form g can be written as a sum of an exact, a coexact and a harmonic k-form: $g = df + d^*h + k$ and that harmonic forms are the intersection of the kernel of d and d^*.

Theorem: $\text{im}(H) = \text{im}(d) + \text{im}(d^*)$, $\ker(H) = \ker(d) \cap \ker(d^*)$.

Proof. The operator $H : \Lambda \to \Lambda$ is symmetric so that image and kernel are perpendicular. The image of H splits into two orthogonal components $\text{im}(d)$ and $\text{im}(d^*)$. That $f = dg$ and $h = d^*k$ are orthogonal is seen from $\langle dg, d^*k \rangle = \langle ddg, k \rangle = 0$.
To the second statement: the kernel of H is contained both in the kernel of d and the kernel of $d*$ because $Hf = 0$ implies $0 = \langle f, Hf \rangle = \langle d^*f, d*f \rangle + \langle df, df \rangle = |d^*f|^2 + |df|^2$, so that both $d^*f = 0$ and $df = 0$. On the other hand, if $df = 0$ and $d^*f = 0$, then $Hf = dd^*f + d^*df = 0$. □

5.7. Let $d_k : \Lambda_k \to \Lambda_{k+1}$ be the restriction of d onto Λ_k. If the kernel $\ker(d_k)$ of dimension z_k and range $\text{im}(d)$ has dimension r_k, the rank-nullity theorem in linear algebra tells $\dim(\ker(d_k)) + \dim(\text{im}(d_k)) = f_k$, we get $z_k = f_k - r_k$.

Theorem: $b_k = \dim(\ker(d_k))/\text{im}(d_{k-1})$ and so $b_k = z_k - r_{k-1}$.

Proof. $\text{im}(d_{k-1})$ is contained in $\ker(d_k)$. The $n \times f_k$ matrix D_k containing the blocks d^*_{k-1} and d_k satisfies $D^*_k D_k = H_k$. Now use that $\text{im}(d_{k-1})$ is the orthogonal complement of $\ker(d^*_{k-1})$ and row reduce F_k to see that b_k, the number of redundant columns in F_k are the columns where no leading 1 for d^*_{k-1} and no leading 1 for d_k appears. □

5.8. Adding $z_k = f_k - r_k$ and $b_k = z_k - r_{k-1}$ gives $f_k - b_k = r_{k-1} + r_k$. Taking the alternating sum on both sides using $r_{-1} = 0$ and $r_k = 0$ for $k > q$, telescopes $\sum_{k=0}^{q} (-1)^k (f_k - b_k) = \sum_{k=0}^{\infty} (-1)^k (r_{k-1} + r_k) = 0$. This is the traditional proof of the Euler-Poincaré theorem.

5.9. Examples.

- The G comes from a graph, the matrix H_0 is the **Kirchhoff matrix**. The number b_0 is the number of connected components.
- The number b_1 counts the number of 1-dimensional holes. For a connected G the number b_q is either 1 or 0 depending on whether G is orientable.
- For connected orientable surfaces, we have $b_0 = b_2 = 1$ so that $\chi(G) = |V| - |E| + |F| = 2 - b_1 = 2 - 2g$ where g is the genus of the surface.
- For connected regions in the plane we have $|V| - |E| + |F| = 1 - g = b_0 - b_1$.
- A contractible graph has $b_0 = 1$ and $b_k = 0$ for all $k > 0$.
- For a q-sphere $b_0 = b_q = 1$ and $b_k = 0$ for all $0 < k < q$.

Oliver Knill, knill@math.harvard.edu, Elements of Finite Geometry, 2026