# ELEMENTS OF FINITE GEOMETRY

OLIVER KNILL

## Unit 4: Euler's Gem

A $q$-sphere $G$ is a $q$-manifold which when punctured becomes contractible. Euler's Gem formula tells that Euler characteristic is $\chi(G) = 1 + (-1)^q$. We can classify all Platonic q-spheres, inductively defined as $q$-spheres for which all unit spheres are isomorphic and equal to some Platonic $(q-1)$-sphere.

**4.1.** A graph $(v, E)$ is **contractible** if there exists $v \in V$ such that $S(v)$ and $G \setminus v$ are both contractible. The 1-point graph $1 = K_1$ is contractible. The empty graph is not contractible.

**4.2.** The empty-graph is the $(-1)$-sphere. A graph $G$ is called a **q-manifold** if all unit spheres $S(v)$ are $(q-1)$-spheres. A $q$**-sphere** is a $q$-manifold for which there is a $v$ such that $G \setminus v$ is contractible.

**4.3.** Every contractible graph has $\chi(G) = 1$ by induction: $\chi(G) = \chi(G \setminus v) + \chi(S(v)) - \chi(\{v\}) = 1 + 1 - 1 = 1$. If $A, B$ are graphs $A \cap B$ carries the intersection of the simplicial complexes of $A$ and $B$, then the **valuation formula** for simplicial complexes holds also for graphs $\chi(A) + \chi(B) - \chi(A \cap B) = \chi(A \cup B)$. $A \cup B$ carries the intersection of the simplicial complexes of $A$ and $B$. The Euler Gem formula is:

**Theorem:** $\chi(G) = 1 + (-1)^q$ for a q-sphere.

*Proof.* (i) In every graph, every unit ball $B(v)$ is contractible. It follows that $\chi(B(v)) = 1$. (ii) for a $q$-sphere, every unit sphere $S(v)$ is a $(q-1)$-sphere and so has by induction the Euler characteristic $1 + (-1)^{q-1}) = 1 - (-1)^q$.
(iii) Now use $\chi(G) = \chi(G \setminus v) + \chi(B(v)) - \chi(S(v)) = 1 + 1 - (1 + (-1)^{q-1}) = 1 + (-1)^q$. We used the valuation formula. □

**4.4.** The **join** of two graphs $A \oplus B$ has vertex set $V(A) \cup V(B)$ and edge set $E(A) \cup E(B) \cup \{(a, b), a \in V(A), b \in V(B)\}$. It is the dual of disjoint union + because $A \oplus B = \overline{\overline{A} + \overline{B}}$. With this operation and the zero element 0 we have a monoid. Spheres form a sub-monoid:

**Theorem:** A is a $k$-sphere, $B$ is a $l$-sphere $\Rightarrow A \oplus B$ is a $(k+l+1)$-sphere.

*Proof.* Use induction with respect to $k + l = n$. For $n = 0$ meaning $k = l = 0$, the join is a 1-sphere. Assume we have proven it for all cases with $k + k \leq n - 1$ and assume $k + l = n$. By definition every unit sphere $S_A(v)$ is a $(k-1)$-sphere and every unit sphere $S_B(w)$ is a $(l-1)$-sphere. For $v \in V(A)$, the graph $S_{A \oplus B}(v) = S_A(v) \oplus B$ is a $l - 1 + k + 1$-sphere by induction. For $w \in V(B)$, the graph $S_{A \oplus B}(w) = A \oplus S_B(w)$ is a $l + k - 1 + 1$-sphere by induction. This shows that $A \oplus B$ is a $l + k + 1$ manifold. Now use that if $A$ is contractible and $B$ is arbitrary than $A + B$ is contractible. This shows that for $v \in V(A)$, the graph $A \oplus B \setminus v = (A \setminus v) \oplus B$ is contractible. Similarly, for $w \in V(B)$, the graph $A \oplus B \setminus w = A \oplus (B \setminus w)$ is contractible. ☐

**4.5.** Inductively, a q-sphere is declared to be **Platonic**, if all unit spheres are Platonic $(q-1)$- spheres that are all isomorphic. There is a unique Platonic sphere in all dimensions except dimensions 1,2,3. There are infinitely many 1-spheres $C_n$ and they are all Platonic. For d=2,there is the **octahedron** and **icosahedron**. For d=3, there is the **600 cell** and the **16 cell**. After that there are only cross polytopes, the $2^{q+1}$ cell.

**Theorem:** There is a unique Platonic sphere for $q > 3$.

*Proof.* $q = -1, 0, 1$ are clear. For $q = 2$, the curvature $K(x) = 1 - f_0/2 + f_1/3 - f_2/3$ is constant, adding up to 2. It is either 1/3 or 1/6. For $d = 3$, where each $S(x)$ must be either the octahedron or icosahedron, $G$ is the 16 cell or 600 cell. For $d = 4$, by Gauss-Bonnet, $K(x)$ add up to 2 and be of the form $L/12$ for some integer. For $L = 1$, there exists the 4-dimensional cross polytope with $f$-vector $(10, 40, 80, 80, 32)$. There is no 4-sphere, for which $S(x)$ is the 600-cell. As the $f$-vector of it is $(120, 720, 1200, 600)$, we would get $K(x) = 1 - 120/2 + 720/3 - 1200/4 + 600/5 = 1$ requiring $|V| = 2$ and $\dim(G) \leq 1$. Once down to 1, we inductively only can have one Platonic graph in higher dimensions. ☐

**4.6.** When classifying Platonic solids, one traditionally adds the dual polytopes which are not q-manifolds as they are triangle free. Classics always also has added the (q+1)-simplices because their boundary simplicial complex are (q-1)-spheres as simplicial complexes. The 24-cell, the units of the Hurwitz quaternions is the only additional polytop. But we have not proven this here. Classically, one defines regular q-polytopes as boundaries of convex regions in $\mathbb{R}^{q+1}$.

**4.7.** Examples:

- Cross polytopes $K_{2,2,\dots 2}$ are the n-fold suspension of 0. $K_{\{\}} = 0, K_{\{2\}} = \overline{K_2} = S^0$, $K_{\{2,2\}} = K_{2,2} = C_4$. $K_{2,2,2} = S_0 \oplus S_0 \oplus S_0 = O$ is the octahedron. $K_{2,2,2,2} = C_4 \oplus C_4$ is the smallest 3-sphere.
- The join of two icosahedra is a 5-sphere. But it is not Platonic.