ELEMENTS OF FINITE GEOMETRY

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Unit 3: Index Expectation

There is an integral geometric link between Poincaré-Hopf and Gauss-Bonnet: curvature can be realized as the expectation of indices. This probabilistic tool links the discrete with the continuum.

3.1. Assume (V, E) is a finite simple graph. Let R be a totally ordered set, like an interval $[a, b]$ or a subset of the integers. Consider now $(\Omega, \mathcal{A}, \mathrm{P})$, **probability space** of locally injective functions from V to R. An example is the set of all possible colorings from V to $\{1, \dots, c\}$, where c is the **chromatic number** and assume that P is the counting measure giving each coloring the same probability. For fixed v, the index $g \to i_g(v)$ is now a **random variable** on Ω.

3.2. The **index expectation** $K(v) = \mathrm{E}[i_g(v)]$ does not involve g any more and defines a "curvature" K. Of course, K depends on the probability space.

Theorem: If $K(v) = \mathrm{E}[i_g(v)]$, then $\sum_{v \in V} K(v) = \chi(G)$.

Proof. Start with Poincaré-Hopf $\sum_{v \in V} i_g(v) = \chi(G)$. Take the expectation of this formula to get $\mathrm{E}[\sum_{v \in V} i_g(v)] = \mathrm{E}[\chi(G)] = \chi(G)$. Fubini allows to write the left hand side as $\sum_{v \in V} \mathrm{E}[i_g(v)] = \sum_{v \in V} K(v)$. □

3.3. Every vertex $v \in V$ defines a random variable $X_v(g) = g(v)$. If these random variables are independent and have a uniform distribution on some interval we say P has the **Lebesgue property**.

Theorem: If P has the Lebesgue property, then K is the Levitt curvature.

Proof. For every k-simplex $x = (x_0, \dots, x_k)$, the probability that a function g has the maximum on x_j is the same as the values of g outside g are independent and so do not factor in. By assumption the situation on the simplex is now symmetric with respect to any permutation of the simplices. This means that each vertex gets $1/(k+1)$ of the energy $\omega(x)$ like in the curvature case. □

3.4. Assume c is the **chromatic number** of G. Let Ω denote the set of all c colorings. If P has the uniform distribution on this finite set, we say P is the **coloring probability space**.

Theorem: For the coloring probability space, K is the Levitt curvature.

Proof. Also this is a symmetry argument. Since the probability measure does not change if we permute the coloring space, each point x_j in a simplex has the same probability of being the maximum . $\square$

3.5. The index itself can be seen as a curvature if P is supported on a single point g in Ω. By changing the probability space, we can deform the geometry. Index expectation allows us to modify the curvature.

3.6. If the graph (V, E) is a triangulation of a compact Riemannian manifold M is isometrically Nash embedded into an ambient Euclidean space E, then almost all linear functions $g(x) = x \cdot a$ are Morse functions. For a sufficiently fine triangulation G in M, almost all linear functions in the ambient space are locally injective on G. The Poincaré-Hopf indices at critical points match up with the classical ones. The index expectation is then a curvature that is locally homogeneous. It has to be the Gauss-Bonnet-Chern integrand, as Weyl already pointed out.

3.7. Curvature is compatible with the Shannon product $A * B$ of two graphs A, B.

Theorem: The curvature $K(x, y)$ on $A * B$ is $K(x, y) = K(x)K(y)$.

Proof. (i) First show $i_{G*H,gh}(x, y) = i_{G,g}(x)i_{H,h}(y)$. Proof: For any graphs G, H, we have $(1-\chi(G))(1-\chi(H)) = (1-\chi(G \oplus H))$, where $G \oplus H$ is the graph join. The formula $(1-\chi(S_{G*H,g*h}(x))) = (1-\chi(S_{G,g}(x)))(1-\chi(S_{H,h}(x)))$ follows now from $S_{G*H,g*h}(x, y)$ being homotopic to the join of $S_{G,g}(x)$ and $S_{G,h}(y)$ and that the Euler characteristic is a homotopy invariant. We have $S^-_{G*H}(x, y) = B^-_G(x) * S^-_H(y) \cup S^-_G(x) * B^-_H(y)$, which is homotopic to the join of S^-_H and S^-_G.
(ii) Take the expectation of the relation in (i) and make use of the fact that the random variables $X(g) = i_{G,g}(x)$ and $Y(h) = i_{H,h}(y)$ are independent. The expectation of the product is the product of the expectations. Since $K(x) = \mathrm{E}[X], K(y) = \mathrm{E}[Y]$, we have $K(x, y) = \mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$. $\square$

3.8. Examples:

(1) If the probability measure is concentrated on one single point g, we get Poincaré-Hopf.
(2) If the probability space is $\prod_{v \in V}[0, 1]$ with the product Lebesgue measure then almost all functions are injective and $K(v)$ is the Levitt measure. The same if we take independent random values in a finite set of numbers and condition on colorings.
(3) Assume G is embedded in $\mathbb{R}^2$, meaning $P(v) = (x(v), y(v))$ is given. Let $g(v) = \cos(\theta)x(v) + \sin(\theta)y(v)$ and $\Omega = \{\theta \in [0, 2\pi)\}$. Index expectation defines a curvature that depends on the embedding.
(4) Since the symmetric index $[i_g + i_{-g}]/2$ is zero for an odd dimensional manifold any probability measure that is invariant under the involution $g \to -g$ produces a curvature that is constant zero on an odd-dimensional manifold.