ELEMENTS OF FINITE GEOMETRY

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Unit 2: Poincare-Hopf

The Poincare-Hopf theorem writes the Euler characteristic of a space as the degree of a divisor, the sum of integer indices attached to points. This works for locally injective scalar function or locally non-circular vector fields.

2.1. Geometry is a finite simple graph (V, E). A **scalar function** g is a map $g : V \to R$, where R is a totally ordered set, like a subset of $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. We assume that g is a **coloring**, meaning that it is **locally injective**: $g(v) \neq g(w)$ if $(v, w) \in E$. For $v \in V$, the **stable sphere** $S_g^-(v)$ is the sub-graph of the unit sphere $S(v)$, generated by $\{w \in S(v), g(w) < g(v)\}$. Define $i_g(v) = 1 - \chi(S_g^-(v))$.

Theorem: $\chi(G) = \sum_{v \in V} i_g(v)$.

Proof. Distribute the energy $\omega(x)$ of each simplex x to the vertex w in x, where g takes its maximum. Then $i_g(v)$ is the total energy which the vertex v has collected. □

2.2. In terms of the **simplex generating function** $f_G(t) = 1 + f_0(G)t + \cdots + f_d(G)t^{d+1}$, one has

Theorem: $f_G(t) = 1 + t \sum_{v \in V} f_{S_g^-(v)}(t)$.

Proof. Every k-simplex x in $S_g^-(v)$ gives its energy $\omega(x)$ to v. The simplex $x + v$ contributes to the count of the $k + 1$ simplex. All energies $\omega(x), x \in G$ are accounted for. □

2.3. This again gives a recursive computation of $f_G(t)$ and so the cliques of G. It is even more efficient than Gauss-Bonnet as $S_g^-(v)$ is in general smaller than $S(v)$. The index of a vertex is

$$i_g(v) = f_{S_g^-(v)}(-1) \; .$$

2.4. A **vector field** is a map $F : G \to V$ with $F(x) \in x$. A **directed graph** with the property that there are no closed loops in each simplex x defines a vector field by $F(x) = \max_{v \in x}(v)$, as the direction defines then a total order on x. We call this a **locally non-circular digraph**.

2.5. Define $S_F^-(v) = \{x \in G, F(x) = v\} = F^{-1}(v)$ and $i(v) = 1 - \chi(S_F^-(v))$. We again have $f_G(t) = 1 + t \sum_{v \in V} f_{S_F^-(v)}(t)$.

2.6. An example is the **gradient field** defined by a locally injective function g. In this case $F(x) = v$, where v is the maximum of g on x. The index of such a graph is then $i(v) = 1 - \chi(S^-(v))$, where $S^-(v) = \{w \in S(v), w < v\}$.

Theorem: For a locally non-circular directed graph, $\chi(G) = \sum_{v \in V} i(v)$.

2.7. Define the **symmetric index** $j_g(v) = [i_g(v) + i_{-g}(v)]/2$. By linearity, we still have $\sum_{v \in V} j_g(v) = \chi(G)$.

2.8. Examples:

(1) Let $G = C_n$ be a cycle graph and g a function. Now $i_g(v) = 1$ for local minima and $i_g(v) = -1$ for local maxima. Since minima and maxima alternate, there are the same number of minima and maxima.

(2) If $G = S_n$ is a star graph, and the maximum is at the center c then $i_g(c) = n-1$ and all leaves have $i_g(v) = 1$. The total of all index values is 1. If $g(c)$ is the k'th maximal entry then $i_g(c) = n-k$ and $k-1$ entries that are smaller produce leaf indices 1.

(3) If v is a local minimum of g on V, then $i_g(v) = 1$ because $S(v) = 0$ is the empty graph which has Euler characteristic 0. If v is a local maximum of g on V, then $i_g(v) = 1 - \chi(S(v))$. For a 2-manifold for example where $S(v)$ is a circular graph we have $i_g(v) = 1$ at a maximum too. At a saddle point, where $S(v)$ consists of two disconnected arcs, we have $i_g(v) = -1$.

(4) If G is a complete graph and g is an arbitrary function, then $i_g(v) = 1$ at the minimum and $i_g(v) = 0$ else. The symmetric index is $1/2$ on the maximum and $1/2$ on the minimum.

(5) If G is a manifold for which $\chi(S(v)) = 2$ for all v like for odd dimensional manifolds, then $\chi_{S^-_{-g}}(v) = 2 - S^-_g$ which can be written as $i_g(v) = -i_{-g}(v)$ so that $j_g(v) = 0$ for all v. Odd dimensional manifolds all have Euler characteristic 0, independent of whether they are orientable.

(6) If G_1 is the Barycentric refinement of G in which the vertices are the simplices of G and two vertices are connected if one is contained in the other, look at the coloring $g(x) = \dim(x)$. Now $S^-_g(x)$ is the boundary complex of the simplex x and $1 - \chi(S^-_g(x)) = \omega(x)$. The Poincaré-Hopf formula now tells that $\chi(G) = \sum_x \omega(x) = \sum_{v \in V(G_1)} i_g(v) = \chi(G_1)$. The Barycentric refinement G_1 has the same Euler characteristic than G.

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