ELEMENTS OF FINITE GEOMETRY

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Unit 1: Gauss-Bonnet

The Gauss Bonnet theorem equates the total curvature of a geometry with its Euler characteristic. Curvature is the energy of the fundamental units of space pushed to points.

1.1. A simple model for geometry is a **finite simple graph** (V, E). Its complete subgraphs define a finite abstract simplicial complex G, a set of non-empty finite sets that is closed under the operation of taking non-empty subsets. The elements in G are known as **simplices**, **cliques** or **faces**. The empty set $\emptyset$ is not a simplex. But the empty set is called the **void** and is a simplicial complex. A simplex x of $(n+1)$ points has **dimension** $\dim(x) = n$. Its **energy** is $\omega(x) = (-1)^{\dim(x)}$. The total energy $\chi(G) = \sum_{x \in G} \omega(x)$ is the **Euler characteristic** (V, E).

1.2. Let $f_k(G)$ denote the number of elements $x \in G$ that have dimension k. For example, $f_0(G) = |V|$ and $f_1(G) = |E|$ and $f_2(G)$ is the number of triangles in the graph. We have $\chi(G) = \sum_{k=0}^{q} (-1)^k f_k(G)$, where q is the **maximal dimension.**

1.3. The **unit sphere** of a vertex v is the sub-graph generated by all its neighbors: $S(x) = (W = \{w \in V, (v,w) \in E\}, \{e = (a,b) \in E, a \in W, b \in W\})$. The **curvature of a vertex** is v is defined as

$$K(v) = 1 - \sum_{k=0}^{q-1} \frac{(-1)^k f_k(S(v))}{k+1} .$$

Theorem: $\chi(G) = \sum_{v \in V} K(v)$.

Proof. Distribute the energy $w(x)$ of each simplex x of dimension k equally to each of its $k+1$ vertices. Then $K(v)$ is the total energy, that v has collected. □

1.4. A different proof is obtained by noting the $\chi(G)$ is a linear combination of valuations $f_k(G)$ which satisfy $f_k(G) = \frac{1}{k+1} \sum_v f_{k-1}(S(v))$ generalizing the **Euler handshake formula** $f_1(G) = \frac{1}{2} \sum_v f_0(S(v)) = \frac{1}{2} \sum_v d(v)$, where $d(v) = f_0(S(v))$ is the vertex degree.

1.5. To generalize the theorem to simplicial complexes G, let V be the 0-dimensional simplices (naturally identified with $\bigcup_{x \in G} x$). The unit sphere is $S(v) = \{x, v \in x, \{v \neq x\}\} = \overline{U(v)} \setminus U(v)$ is a simplicial complex for which $f_k(S(v))$ is defined. Curvature is supported on V.

1.6. The **simplex generating function** $f_G(t) = 1 + f_0(G)t + \cdots + f_d(G)t^{q+1}$ of G satisfies $\chi(G) = 1 - f_G(-1)$. The derivative of a polynomial can be defined without taking limits by using the rule $f' = \sum_{n=0}^{q} a_n n x^{n-1}$ if $f = \sum_{n=0}^{q} a_n x^n$. Gauss-Bonnet is now an elegant recursive formula:

Theorem: $f'_G(t) = \sum_{v \in V} f_{S(v)}(t)$.

Proof. Use the Gauss-Bonnet theorem and check that $K(v) = \int_{-1}^{0} f_{S(v)}(t)\, dt$. □

Examples:

(1) If G is 1-dimensional, there are no triangles and and $\chi(G) = f_0(G) - f_1(G) = |V| - |E|$. The curvature is $K(v) = 1 - \deg(v)/2$, where $\deg(v)$ is the **vertex degree**. **Circular graphs** C_n for example have curvature constant 0. **Star graphs** S_n have curvature 1/2 on the n leafs and curvature $1 - n/2$ at the branch center. The total curvature of a star graph is 1. **Path graphs** P_n have curvature 1/2 at the boundary and 0 else. The **cube graph** C has constant curvature $K(v) = 1 - 3/2 = -1/2$ leading to $\chi(C) = -4$. The **dodecahedron graph** D has $K(v) = 1 - 3/2 = -1/2$ and $\chi(D) = -10$.

(2) A graph is called a 2-**manifold** if every unit sphere $S(v)$ is a circular graph of length ≥ 4. The Euler characteristic is then $\chi(G) = |V| - |E| + |F|$. An example is the octahedron O with $|V| = 6$ vertices or the icosahedron I with $|V| = 12$ vertices. The curvature of a 2-dimensional manifold is the **Eberhard formula** $1 - \deg(v)/6$ because $f_0(S(v)) = f_1(S(v)) = \deg(v)$ and $f_0(S(v))/2 - f_1(S(v))/3 = \deg(v)/6$. For the octahedron, the curvature is constant $K(v) = 1 - 4/6 = 1/3$ adding up to $\chi(G) = 2$. For the icosahedron, the curvature is constant $K(v) = 1 - 5/6 = 1/6$ adding up to $\chi(G) = 2$.

(3) A complete graph K_n has n vertices and constant $K(v) = 1/n$. One can see this by noting that the curvature must be constant and add up to 1 and that the Euler characteristic of K_n is 1. Note that $f_k(K_n) = \binom{n}{k}$ so that $(1+t)^n = \sum_{k=0}^{n} f_{k-1}(K_n) t^k$ and for $t = -1$, this is 0 to the left and $-1 + \chi(K_n)$ on the right. For the **tetrahedron** $T = K_4$ for example, the curvature is constant 1/4. It is not a 2-manifold.

(4) The 4-partite graph $K_{2,2,2,2}$ is the **3-dimensional cross polytope**. It is a discrete **3-sphere**. Its f-vector is $f = (f_0, f_1, f_2, f_3) = (|V|, |E|, |F|, |C|) = (8, 24, 32, 16)$, where F is the set of triangles, the 2-dimensional parts of space and C is the set of tetrahedra, the 3-dimensional parts of space. The Euler characteristic is $\chi(G) = 8 - 24 + 32 - 16 = 0$, like for any odd-dimensional manifold. The curvature is constant 0.