ELEMENTS OF FINITE GEOMETRY

OLIVER KNILL

Unit 0: Introduction

There is a fundamental possibility that the common axiom system used by mathematicians is inconsistent. A strong enough foundation of mathematics like the Zermelo-Frenkel frame work can never be proven to be consistent within itself and is always extendable in the sense that there are statements that can not be proven nor dis-proven within the system. Hilbert's dream of building a provably consistent strong theory was a pipe dream. The incompleteness theorems have entered popular culture especially with Hofstadter's book "Goedel-Escher-Bach". While most mathematicians - including myself - do not worry about a possible inconsistency dooms-day, the possibility of such an event is a motivation to pursue geometry using weaker tools. Working with finite mathematics is such a way. Finite descriptions of spaces like manifolds have already been initiated in Poincaré's analysis situs program, but most topologists still assume the continuum. Even Dehn and Sommerville who initiated abstract simplicial complexes in 1907 used the continuum.

Finite mathematics is not weak enough to escape Goedel. But it is conceivable that ZF is inconsistent while an axiom system without infinity remains consistent. When working in a finite geometric set, we can in principle be certain to have consistency within that world. Most computer scientist are by nature even more radical. We are "strict finitists" in the sense that we only work with structures for which there are tools to handle them. We can make sense of the symbol $GP = 10^{(10^{100})}$, but not of the number "googolplex" itself. As a computer scientist, I am essentially blind for answering questions like how many primes there are in $[GP - 100, GP + 100]$. Working with finite structures puts mathematics on safer grounds. It is a refuge to which we might have to retreat one day, if some inconsistency in ZF should emerge. I personally think this can happen but that there is little to worry: finite tools are perfectly able to emulate continuum math. It is still possible that even after removing the infinity axiom, a catastrophic collapse could occur, but that is much less likely; it would mean that even the algebra of finite sets is inconsistent. Assuming consistency of finite systems requires some faith, similarly as we trust the Church thesis or assume our brain memory is reliable, allowing us to do logical steps without forgetting the assumptions, while doing so. In the last couple of years we all have seen intrinsically random probabilistic thinking entities to emerge that have started to compete with human thought. Aristotle would turn in his grave learning that humanity started to flip coins to do thinking. Probabilistic reasoning is good enough for practical purposes but philosophically unacceptable even if the probability to be wrong is only $1/GP$.

A "finite geometry" is a finite geometric structure like a simplicial complex, a graph or a delta sets. These combinatorial structures carry natural arithmetic and topological operations. One can add or multiply such structures in various ways for example. The choice of using these three structures a bit arbitrary but it is motivated by the Unix trilogy "simplicity, clarity and generality": simplicial complexes have only one axiom and so are simple, graphs are intuitive and visual and so very clear. Delta sets finally are the most general. Every graph defines a simplicial complex and every simplicial complex is naturally a delta set. Delta sets are sets of sets that admit a calculus structure. Their advantage of the delta set category is that has no limitations but still allows to do all geometry that we know from in the continuum. For example: submanifolds or products of geometric structures are often delta set at first before modeled again as a graph. Multi-graphs like quivers are delta sets. We work in this trinity.

It is tempting to label a discussion about finite geometry with terms like "quantum geometry". The meaning of the word "quantum" has however been washed out and not only has become ambiguous, even misleading. The term "quantum gravity" illustrates it. Classical geometry or calculus can be "quantized" in various ways: there are non-commutative geometry flavors, Poisson brackets can be replaced by commutators or Lagrange variational problem are deformed to path integral quantization. Every numerical scheme is some sort of quantization, as the continuum is modeled using finite structures suitable for numerical purposes.

Finite geometry is related to "calculus without limits". For example, every function f on the real line satisfies the exact Taylor formula $f(a+x) = \sum_{n=0}^{\infty} f^{(n)}(a)x^n/n!$, for integer x, if the derivative is deformed to $Df(a) = f'(a) = f(a+1) - f(a)$ and the polynomial algebra is deformed to $x^n = x(x-1)\dots(x-n-1)$. No regularity whatsoever is needed and Taylor is a finite sum. For $x = 2$ for example $f(2) = f(0)+f'(0)2/1!+f''(0)2(2-1)/2! = f(0)+2f(1)-2f(0)+(f(2)-2f(1)+f(0)) = f(2)$. If the unit 1 is the Planck scale, finite calculus produces the same physics as traditional calculus. The derivative D generates translation because $\exp(Dt)f(x) = f(x+t)$ and $Xf(x) = xf(x-1)$ and leads with the momentum operator $P = iD$ to the canonical anti-commutation relations $[X, P] = (XP - PX) = i$.

Theorems are at the heart of mathematics. A theory needs to be judged by the quality of theorems it produces. Ideally, definitions are short and clear, theorems simple and proofs intelligible. In a first batch we cover 12 results:

- Gauss Bonnet
- Poincare Hopf
- Index expectation
- Euler's Gem
- Euler Poincare
- Unimodularity
- Brouwer-Lefschetz
- Sphere formula
- Level sets
- Index formula
- Quadratic cohomology
- Higher characteristics

Oliver Knill, knill@math.harvard.edu, Elements of Finite Geometry, 2026