A SUMMER OF GEOMETRY

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Appendix 3: Linear algebra

3.1. The **adjacency matrix** of a finite simple graph Γ with n vertices and m edges is a $n \times n$ matrix $A(x,y) = 1$ if $(x,y) \in E$ and $A(x,y) = 0$ else. For oriented graphs, the adjacency matrix is $A(x,y) = 1$ if $(x,y) \in E$. The adjacency matrix of a digraph is no more self adjoint in general. For quivers it is custom to write $A(x,y) = k$ if there are k connections and $A(x,x) = k$ if there are k loops.

3.2. The **Kirchhoff matrix** or **Kirchhoff Laplacian** K of a graph Γ of n vertices is $K = D - A$, where $D = Diag(d_1, \dots, d_n)$ is the diagonal matrix with vertex degrees $d_i = d(v_i)$. One can write $d(v) = f_0(S(v))$, the number of vertices in the unit sphere of v.

3.3. In order to define the exterior derivative, we assume a simplicial complex to be ordered in the sense that every simplex $x \in G$ is given a particular order. As no compatibility is required, the ordering can be arbitrary. If $x \subset y$, the order of y does not have to inherit the order of x. We usually assume that V is ordered in an arbitrary way and that $x = (x_0, \dots, x_k)$ is ordered according to the order in V. The order of G does not matter either, but we can just order it lexigographically. This will assure that the matrices we are going to define are block matrices. If G is a set of non-empty sets, we can achieve all this by initially sorting G as such.

3.4. The **exterior derivative matrix** d of a complex G is $d(x,y) = \text{sign}(x, (x \setminus y)y)$ if $\dim(y) = \dim(x) - 1$ and $y \subset x$. If an other order is used and every x is permuted to x' then this produces a coordinate change UdU^*, where U is unitary with diagonal entries $U(x,x) = \text{sign}(x,x')$. The **Dirac matrix** is $D = d + d^*$. The **Hodge Laplacian** is $H = D^2$.

3.5. Given a simplicial complex G, we can look at the **connection matrix** $L(x,y) = 1$ if $x \cap y$ is non-empty and $L(x,y) = 0$ if $x \cap y$ is empty. This matrix is unimodular and the Fredholm matrix $L = 1 + A$ of the adjacency matrix A of the connection graph, which has the simplices of G as vertices and connects two if they intersect in a non-empty set. Since L can have negative and positive eigenvalues and should be seen as a connection analog of the Dirac matrix as the Dirac matrix is based on inclusion, not intersection. The matrix L^2 is a positive definite Laplacian.

3.6. Given a simplicial complex G, the dimension function filtrates it and $l^2(G)$ is the direct sum of $l_2(G_i)$. Given an operator $A : l^2(G) \to l^2(G)$ that preserves the spaces $L^2(G_i)$ and induces there maps A_i, the **trace** $\mathrm{tr}(A)$ is equal to $\sum_{i=0}^q \mathrm{tr}(A_i)$. Define the **super trace** as $\mathrm{str}(A) = \sum_{i=0}^q (-1)^i \mathrm{tr}(A_i)$. The super trace of the identity matrix 1 is $\chi(G)$. The super trace of H^n is 0 if H is the Hodge Laplacian and n is positive. The super trace of the connection Laplacian L as well as the super trace of its inverse g are both equal to the Euler characteristic $\chi(G)$.

3.7. Here is a lemma that is useful in Hodge theory. The block matrix $D_k = [0, d_{k-1}^*, 0, d_k, 0]^*$ are the k-form columns of the Dirac matrix D. They produce the k-form Laplacians $L_k = D_k^* D_k$ which has the same kernel than D_k. Since the image of d_{k-1} is contained in the kernel of d_k. This means that the kernel of d_{k-1}^* is perpendicular to the kernel of d_k. This implies that the dimension of the kernel of D_k is the same than the dimension of the space $\ker(d_k)/\mathrm{im}(d_{k-1})$. This is the key of moving the equivalence description of k-th cohomology to a description about the kernel of a concrete matrix L_k. It is easier to talk about concrete vectors than to talk about equivalence classes of vectors.

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s=RandomGraph[{10,20}];
A=Normal[AdjajencyMatrix[s]]
K=Normal[KirchhoffMatrix[s]]
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