A SUMMER OF GEOMETRY

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Appendix 2: Simplicial complexes

2.1. A **finite abstract simplicial complex** G is a finite set of non-empty sets closed under the operation of taking non-empty subsets. This is a structure with one axiom, the **hereditary statement**. As custom in set theory, sets contain different elements. $G = \{\{1,2\},\{1\},\{2\}\}$ is an example of a simplicial complex. The structure $G = \{\{1,1\},\{1\}\}$ would be identified with $\{\{1\},\{1\}\}$ and so with $\{\{1\}\}$. A set does not record multiplicities. If this is needed, like for multi-graphs, we would use sequences. The sequence $\{\{1\},\{2\},\{1,2\},\{1,2\}\}$ for example is a quiver with two edges connecting 1 with 2.

2.2. An equivalent definition is often seen in the literature that involves the **vertex set** $V = \bigcup_{x \in G} x$. The assumption is then that G is a subset of $2^V \setminus \{\{\}\}$ that does not contain the empty set and satisfies the heriditary axiom. Since V can be derived from G, the earlier definition is slightly simpler and equivalent.

2.3. More general than a simplicial complex is a set of sets. This structure has also been called a **hypergraph**. A hypergraph in general does not allow a reasonable calculus. A hypergraph also is allowed to contain the empty set. In order for the theory to work properly, it is important not to let the empty set to be in a simplicial complex. The heriditary assumption is needed to get a notion of derivative that works.

2.4. There are flavors of combinatorial topology that include $\emptyset$ into G. The **Euler characteristic** $\chi(G) = \sum_{x \in G} \omega(x)$ with $\omega(x) = (-1)^{\dim(x)}$ for example would not work. [1] There are compelling reasons to keep $\chi(G)$ the Euler characteristic like compatibility with addition (disjoint union) and multiplication (Cartesian product). There are compelling reasons to keep simplices to be contractible objects for which the closure $\overline{\{x\}}$ is a simplicial complex of Euler characteristic 1 and that spheres are non-contractible objects of Euler characteristic $1 + (-1)^q$. The void is the (-1) sphere. It has Euler characteristic 0 and is not contractible.

2.5. Also in **matroid theory**, there are flavors of the theory that include the empty set and versions, where the empty set is excluded. If the empty set $\emptyset$ is excluded in a matroid G, one can say a matroid is a simplicial complex which additionally satisfies the **exchange axiom**: if $|y| < |x|$ then some $v \in x \setminus y$ satisfies $y \cup \{v\} \in G$. An example of a simplicial complex that is not a matroid would be the path complex $G = \{\{1,2,3\},\{1,2\},\{2,3\},\{1\},\{2\},\{3\}\}$. Since matroid theory had been developed

[1] Extended Euler characteristic $\chi(G) - 1$ is popular as it renders the join multiplicativ.

however in the context of linear algebra first, one usually assumes that a matroid contains the empty set.

2.6. Simplicial complexes can be defined from other structures. Any graph defines a **Whitney complex**, in which the vertex sets of complete subgraphs are the sets. It is also called the **flag complex** or **order complex**. The empty graph generates the void. It is not a complete complex as a simplicial complex never contains the empty set. An other complex is the **forest complex**, in which the edges of a forest are the sets. A forest decomposes into trees and trees are assumed in this set-up to be one-dimensional, meaning to have maximal dimension 1. Single points = seeds, are not yet considered trees. There are frameworks like the matrix forest theorem where it is helpful to include seeds as trees.

2.7. Simplicial complexes can be derived from already given simplicial complexes. The intersection of two simplicial complexes is a simplicial complex. If two simplicial complexes are disjoint, then their intersection is the void, which is a simplicial complex. The union of two simplicial complexes $G = A \cup B$ is a simplicial complex. Also this is clear from the definition. If $x \in G$, then either $x \in A$ or $x \in B$ so that the hereditary axiom holds. If G is a simplicial complex of dimension q, the **skeleton complex** of dimension $k \leq q$ is the set of sets for which the dimension is smaller or equal than k.

2.8. Given an arbitrary set A of non-empty sets, one can look at the closure $\overline{A}$ of A. It is the smallest simplicial complex containing A. To get $\overline{A}$, just adjoin all the non-empty subsets of A. The name closure is adequate because it is the closure operation in the Alexandroff topology, the topology first considered by Pavel Alexandrov. [2]

2.9. The set of finite abstract simplicial complexes forms a category in which the objects are the simplicial complexes and the morphisms are simplicial maps $f : A \to B$. A **simplicial map** maps $V(A)$ to $V(B)$ and preserves the order. If $x \subset y$, then $f(x) \subset f(y)$. If f is invertible and the inverse is a simplicial map, then we have a simplicial isomorphism. Maximal dimension can only decrease under simplicial maps. A version of Cantor-Schröder-Bernstein assures that if there are injective simplicial maps from A to B and from B to A, then A, B are isomorphic. The injectivity assumption assures that $V(A), V(B)$ have the same cardinality. The bijection then defines a bijection from A to B as simplices are mapped to simplices.

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Generate[A_]:=Sort[Delete[Union[Sort[Flatten[Map[Subsets,Map[Sort,A]],1]]],1]];
Whitney[s_] :=Map[Sort,Generate[FindClique[s,Infinity,All]]];
EulerChi[G_]:=Sum[-(-1)^Length[G[[k]]],{k,Length[G]}];
Fvector[s_]:=Delete[BinCounts[Map[Length,Whitney[s]]],1];

(* recursive inductive computation of the simplicial complex of a graph *)
UnitSphere[s_, v_]:=VertexDelete[NeighborhoodGraph[s, v], v];
q[X_, v_]:=Table[Sort[Append[X[[k]], v]],{k,Length[X]}];
W[s_]:=Module[{V,A,n},V=VertexList[s];n=Length[V]; A=Table[{V[[k]]},{k,n}];
 If[n==0,{},Union[A,Flatten[Table[q[W[UnitSphere[s,V[[k]]]],V[[k]]],{k,n}],1]]]];
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[2] Alexandrov is the common spelling for the person, Alexandroff is the common use for mathematical results.