

A SUMMER OF GEOMETRY

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Appendix 1: Sets

1.1. Finite sets are an elementary topos that extends to a commutative ring for which the cardinality function produces a ring homomorphism to the integers. Every finite set G of sets also defines a Boolean ring in which the symmetric difference is the addition and the intersection is the multiplication. The zero element is the empty set, the 1-element is the union $V = \bigcup_{x \in G} x$.

1.2. A **finite set** is a collection G of finite objects. It is a standard assumption in set theory that a set does not record multiplicities. The object $\{a, a, b, a, b\}$ for example would be identified with $\{a, b\}$. The former could be modeled as a multi-set or function $\{a, b\} \rightarrow \mathbb{N}$ encoding the multiplicities or a sequence $\{a, a, b, a, b\}$, where the order of appearance matters.

1.3. If V is a finite set, $G = 2^V$ is the set of all subsets of V . This set has a **partial order structure** $A \leq B$ meaning $A \subset B$. It is also standard in set theory not to assume the subset operation to be strict. The statement $A \subset A$ is always true. The set 2^V contains the empty set \emptyset . If $V = \{\} = \emptyset$ is the empty set, then $2^V = \{\emptyset\} = 1$ is a set with one element. Let $|A|$ denote the **cardinality** of the set. The cardinality is a non-negative integer $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. We would write \mathbb{N}^+ for the positive integers. We have $|0| = 0, |1| = 1$ and $|2^A| = 2^{|A|}$ justifying the notation. There is a natural identification of 2^{A+B} with the Cartesian product $2^A \times 2^B$ because if x is a subset of $A + B$, then x defines (a, b) with $a = x \cap A$ and $b = x \cap B$. But $(a, b) \in 2^A \times 2^B$.

1.4. Finite sets form a **category** FinSet in which the finite sets are the **objects** and morphism maps $A \rightarrow B$ are the arrows. The composition of arrows is the composition of functions. The isomorphisms in FinSet are the bijective maps. The empty set 0 is the **initial object**, the singleton set 1 is the **terminal object**. The category FinSet is an **elementary topos**. It has **finite limits**, **exponentials** A^B satisfying $A^{B+C} \sim A^B \times A^C$ and a **subobject classifier** essentially given by the characteristic function $x \rightarrow 1_A(x) \in \{0, 1\}$.

1.5. There is an operation on the objects of FinSet . It is the disjoint union $+$, that is also called the **coproduct**. This defines a **monoid structure** in which the void is the zero element. This monoid can be group completed to become a group. One has then not only sets but “negative sets”. There is also a multiplication $A, B \rightarrow A \times B$ called the **Cartesian product**. $A \times B$ is defined as the set of all ordered pairs $\{(a, b), a \in A, b \in B\}$.

1.6. The coproduct can be **group completed** with negative sets, to get a additive group. The category `FinSet` so naturally extends to a commutative ring in which 1 is the one-element and where 0 is the zero-element. If the cardinality function is extended to negative sets as $|-A| = -|A|$, the cardinality function defines a homomorphism from the ring of finite sets to \mathbb{Z} . If sets of the same cardinality are identified, then this is an isomorphism.

1.7. A set of sets G that is closed under intersection $A \cap B$ and symmetric difference $A+B = (A \cup B) \setminus (A \cap B)$ and contains $1 = V$ is a **Boolean algebra**. It is a commutative ring and is called the **Boolean ring** $(V, +, \cdot, 0, V)$, where $0 = \emptyset$ is the zero-element and V is the one-element. The related structure $(V, \cup, \cap, 0, V)$ is a **Boolean lattice**.

1.8. A set of non-empty sets is called a **simplicial complex** if it is closed under the operation of taking non-empty subsets. An arbitrary set of sets G generates a simplicial complex by including all non-empty subsets of sets in G . The empty set \emptyset is a simplicial complex called the **void** but it is never assumed to be an element in any simplicial complex.

```

Union[{1,2,3},{2,3,4}]
SubsetQ[{1,2,3},{1,2,3}]
EmptyGraphQ[Graph[{}]]
Zero=Range[0]; One=Range[1];
SetProduct[A_,B_]:=Table[{A[[k]],B[[1]]},{k,Length[A]},{1,Length[B]};
SetProduct[Range[3],Range[4]]
SetAdd[A_,B_]:=VertexList[GraphDisjointUnion[Graph[A,{}],Graph[B,{}]]];
SetAdd[{1,2,3},{1,2,3,4}]
SetPower[A_,k_]:=Tuples[A,k]

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