7/11/2024 FIRST HOURLY **Maths 21a, O.Knill, Summer 2024**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed. Fill in the boxes

1) **T** F $(\vec{v} \times \vec{w}) \times (\vec{w} \times \vec{v}) = \vec{0}$ for all vectors $\vec{v}$ and $\vec{w}$.

Solution:

Yes, write it as $\vec{u} \times \vec{u}$ with $\vec{u} = \vec{v} \times \vec{v}$.

2) T **F** Whenever the jerk $\vec{r}\,'''(t)$ at $\vec{r}(t)$ is non-zero, then the curvature is non-zero at $\vec{r}(t)$.

Solution:

We can drive on a line and have non-zero jerk.

3) T **F** The vector projection $\vec{P}_{\vec{w}}(\vec{v})$ of a vector $\vec{v}$ onto a vector $\vec{w}$ satisfies $\vec{P}_{2\vec{w}}(\vec{v}) = 2\vec{P}_{\vec{w}}(\vec{v})$.

Solution:

The projection is independent of the length of $\vec{w}$.

4) **T** F If the vectors $\vec{v}, \vec{w}$ have integer components, then the vector $\vec{v} \times \vec{w}$ has integer components.

Solution:

The formula shows this. We never divide.

5) T **F** The domain of the function $f(x, y) = 1/(x^2 + y^2)$ is the entire plane.

Solution:

We do not count $(0, 0)$ as part of the domain.

6) **T** F If $\vec{u}, \vec{v}$ have the same length, then $\vec{u} + \vec{v}, \vec{u} - \vec{v}$ are perpendicular.

Solution:

Multiply out.

7) T F The identity $|\vec{v} \times \vec{w}| \leq |\vec{v}||\vec{w}|$ holds for all vectors $\vec{v}, \vec{w}$.

Solution:

Follows from the fact that the left hand side is the right hand side times $\sin(\alpha)$.

8) T F If $\vec{r}(t)$ is a curve for which $|\vec{r}\,'(t)| = 1$ for all t, then the curvature satisfies $\kappa(t) \leq |\vec{r}\,''(t)|$.

Solution:

This follows from the explicit formula for the curvature.

9) T F It is possible to intersect a cone with a sphere to get a parabola.

Solution:

The intersection is bounded but a parabola is not bounded.

10) T F The set of points in space which satisfy $z^2 - y^2 - 1 = x^2$ is a one sheeted hyperboloid.

Solution:

It is two sheeted

11) T F The projection satisfies $\vec{P}_{\vec{w}+\vec{u}}\vec{v} = \vec{P}_{\vec{w}}\vec{v} + \vec{P}_{\vec{u}}\vec{v}$.

Solution:

Assume for example that $\vec{v}$ is perpendicular to $\vec{w}$. Then the second is parallel to $\vec{u}$ while the first is parallel to $\vec{v} + \vec{w}$.

12) T F If $\vec{v}(t), \vec{w}(t)$ are curves, $(\vec{v} \cdot \vec{w})'$ is $\vec{v}' \cdot \vec{w} + \vec{v} \cdot \vec{w}'$.

Solution:

This is the product rule.

13) T F The surface given in spherical coordinates as $\rho^3 \sin^3(\phi) = 1$ is a cylinder.

Solution:

It translates as $r^3 = 1$ meaning $r = 1$.

14) T F If A, B, C are three different points space then $\vec{AB} \times \vec{AC}$ is never zero.

Solution:

It can be zero if the three points are on a line.

15) T F The line $\vec{r}(t) = [t, t, t]$ intersects the line $\vec{s}(t) = [t, -t, 0]$ in a perpendicular way.

Solution:

The lines go through (0,0,0) and form a right angle.

16) T F The curve given in polar coordinates as $r\cos(\theta) = 1$ is a line.

Solution:

Indeed, $x = r\cos(\theta)$.

17) T F If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (1, \pi/2, \pi)$, then its rectangular Euclidean coordinates are $(x, y, z) = (0, 0, 1)$.

Solution:

It has to be on the y-axes.

18) T F The distance of the points given in polar coordinates as $(r, \theta) = (1, 0)$ and $(r, \theta) = (1, \pi/2)$ is $\sqrt{2}$.

Solution:

These are the points $(1, 0)$ and $(0, 1)$.

19) T F The surface given in spherical coordinates as $\rho \cos(\phi) = 1$ is a cylinder.

Solution:

It means $z = 1$.

20) T F If $\vec{u}, \vec{v}$ form an acute angle and $\vec{u}, \vec{w}$ form an acute angle, then $\vec{v}, \vec{w}$ form an acute angle.

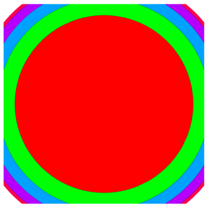
Solution:

Take three vectors in the same plane for which the angles are close to 90, then the other angle is close to 180.

Problem 2) (10 points) No justifications. 0,1,2,3 appear once in a),b),c),d),e)

a) (2 points) Match functions g with their xy-contour plots. Enter 0 if there is no match.

1



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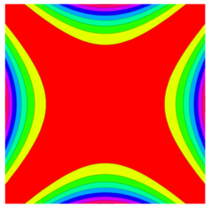
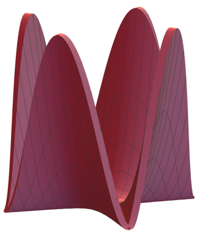


3

Function $g(x,y) =$	0,1,2, or 3
$(x-y)^2$	
$(y-1)^2$	
$(x^2-y^2)^2$	
$(x^2+y^2)^2$	

b) (2 points) Match the graphs of the functions $f(x,y)$. Enter 0 if there is no match.

1



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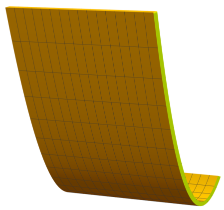

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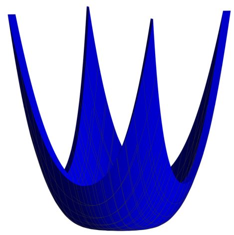
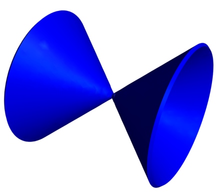


Function $f(x,y) =$	0,1,2, or 3
$(x-y^2)^2$	
$(y-1)^2$	
$(x^2-y^2)^2$	
$(x^2+y^2)^2$	

c) (2 points) Match the surfaces $g(x,y,z) = c$. Enter 0 if there is no match.

1



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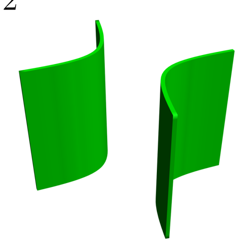

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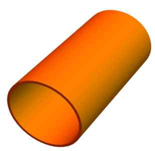
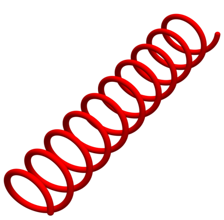


Function $g(x,y,z) =$	0,1,2, or 3
$x^2-y^2=1$	
$x^2+z^2=1$	
$x^2+y^2+z^2=1$	
$x^2-y^2-z^2=0$	

d) (2 points) Match the space curves with the parametrizations. Enter 0 if there is no match.

1



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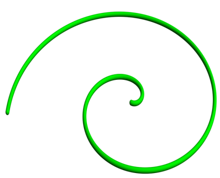

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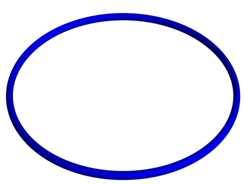
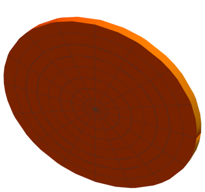


Parametrization $\vec{r}(t) =$	0,1,2, or 3
$[\cos(t),\sin(t),t]$	
$[t\cos(t),t\sin(t),t]$	
$[t\cos(t),t\sin(t),0]$	
$[\cos(t),\sin(t),0]$	

e) (2 points) Match the parametrized surfaces. Enter 0 if there is no match.

1



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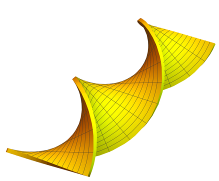

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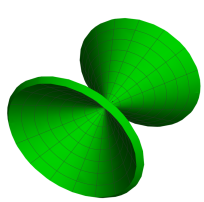


Parametrization $\vec{r}(u,v) =$	0-3
$[u\cos(v),u,u\sin(v)]$	
$[u\cos(v),v,u\sin(v)]$	
$[v\cos(v),u,v\sin(v)]$	
$[u\cos(v),0,u\sin(v)]$	

Solution:

a) 3021 b) 0213 c) 2301 d) 1023 e) 3201

Problem 3) (10 points) Except for c) give computations

We use the notation $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ introduced by **William Hamilton.**

a) (2 points) Find $\vec{i} \times (\vec{i} + \vec{j})$.

Answer:

b) (2 points) What is the angle between $\vec{i}$ and $-\vec{i}$?

Answer:

c) (2 points) Give a vector which does not have a direction.

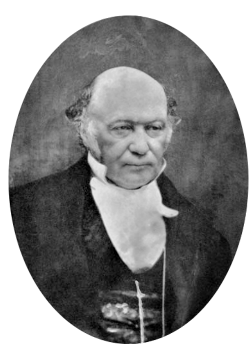
Answer:

d) (2 points) What is the triple scalar product $[\vec{i}, \vec{k}, \vec{j}]$ between $\vec{i}, \vec{k}$ and $\vec{j}$.

Answer:

e) (2 points) What is $(\vec{i} \times \vec{j}) \times (\vec{i} \times \vec{k})$?

Answer:



William Rowan Hamilton, who discovered quaternions.

Solution:

a) $\vec{z}$ b) π c) $\vec{0}$ d) -1 e) $\vec{i}$.

Problem 4) (10 points)

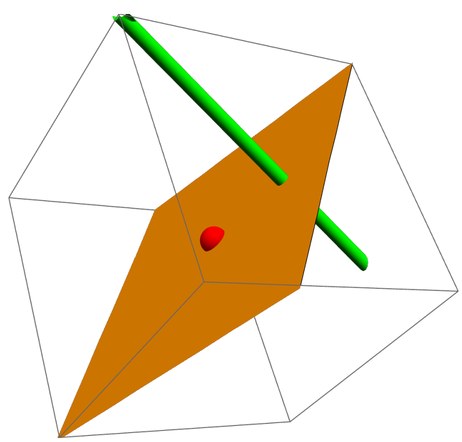
a) (4 points) Find the equation of the plane that passes through the point $A = (1, 1, 2)$ and that is perpendicular to the line $\vec{r}(t) = [1 + t, 2t, 3t]$.

b) (3 points) Verify that the points $B = (0, 0, 3)$ and $C = (9, 0, 0)$ are on the plane.

c) (3 points) Parametrize the plane you found in a).

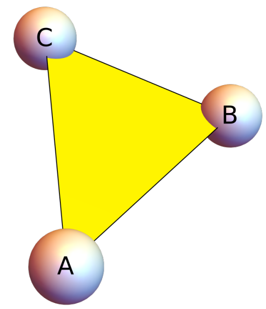
Solution:

a) $x + 2y + 3z = 9$
b) $0 + 2 \cdot 0 + 3 \cdot 3 = 9$ $9 + 2 \cdot 0 + 3 \cdot 0 = 9$ c) $\vec{r}(t) = [1, 1, 2] + t\vec{AB} + s\vec{AC}$.



Problem 5) (10 points)

a) (5 points) What is the distance between the point $C = (1, 2, 3)$ and the line passing through $A = (2, 3, 4)$ and $B = (3, 4, 5)$?



b) (5 points) What is the area of the triangle ABC?

Solution:

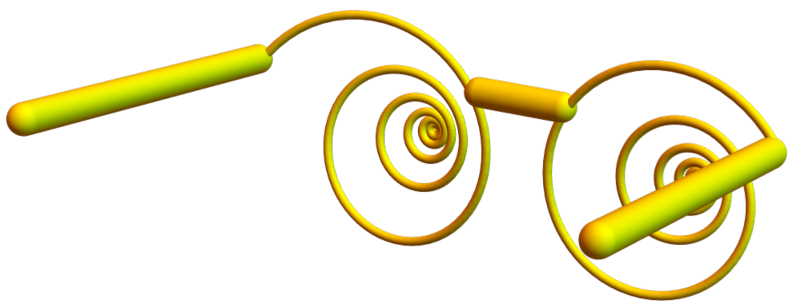
a) $d = |\vec{AB} \times \vec{AC}|/|\vec{AB}| = 0$.
b) $A = |\vec{AB} \times \vec{AC}|/2 = 0$.

Problem 6) (10 points)

We design some fancy glasses in which the frame curls into the glasses. One of the parts is given by the curve

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} e^t \\ e^t \cos(10t) \\ e^t \sin(10t) \end{bmatrix}$$

if $t \in [-\log(10), \log(10)]$ where $\log(x) = \ln(x)$ is the natural log. Find the arc length.



Solution:

The velocity is $\vec{r}\ '(t) = e^t[1, 10\cos(10t) - \sin(10t), 10\sin(10t) + \cos(10t)]$ has length (=speed) $e^t\sqrt{1+1+100}$ because $(10\cos(10t) - \sin(10t))^2 + (10\sin(10t) + \cos(10t))^2 = 1 + 100$. So, $L = \sqrt{102}\int_{-\log(1)}^{\log(10} e^t\, dt = \sqrt{102}(10 - 1/10) = \sqrt{102} \cdot 99/100$.

Problem 7) (10 points)

We FPV fly through the Elliot house bell tower and feel an acceleration

$$\vec{r}\,''(t) = \begin{bmatrix} \cos(t) \\ t^2 \\ -10 \end{bmatrix}$$

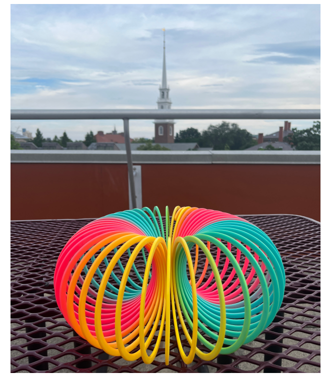
with initial velocity $\vec{r}\,'(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and initial position $\vec{r}(0) = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$. Determine the path $\vec{r}(t)$ of the First Person View drone.

Solution:

Integrate twice $\vec{r}\,'(t) = [-\sin(t), t^3/3, -10t] + [1, 0, 1]$ $\vec{r}(t) = [-\cos(t) + t, t^4/12, -5t^2 + t] + [2, 0, 7]$

Problem 8) (10 points)

The curve $\vec{r}(t) = [3\cos(t), 3\sin(t), 4t]/5$ is the **slinky curve**.
It satisfies $|\vec{r}\,'(t)| = 1$ for all t.



We bent a slinky to be on $\rho = \sin(\phi)$ and photographed it at Harvard.

a) (5 points) Compute the curvature

$$\kappa = |\vec{r}\,'(0) \times \vec{r}\,''(0)|$$

of the curve at $\vec{r}(0)$.
b) (5 points) The number

$$\tau = \frac{[\vec{r}\,'(0), \vec{r}\,''(0), \vec{r}\,'''(0)]}{\kappa^2}$$

involves the triple scalar product of the first three derivatives and the curvature. It is called the **torsion** of the curve at the point $\vec{r}(0)$. Compute it!

Solution:

For heaven's sake do not compute the cross or triple product with the full expressions containing t. Also good is to keep the factor 1/5 outside. Compute $\vec{r}'(0) = [0, 3, 4]/5$ and $\vec{r}\,''(0) = [-3, 0, 0]/5$ and $\vec{r}\,'''(0) = [0, -3, 0]/5$. a) The curvature is the length of $[0, -12, 9]/25$ which is $3/5$. b) The triple scalar product which is $36/125$. Divide this by $\kappa^2 = 9/25$ to get $4/5$.

Problem 9) (10 points) No justifications are needed.

We are charged to do some art work in physics like **orbital states** in an atom or **density level surfaces** near an accretion disk of a black hole. We practice here.

a) (2 points) A first ellipsoid $x^2+(y-3)^2/9+z^2=1$ is parametrized using spherical coordinates. Complete the rest

$$\vec{r}(\theta,\phi) = [\sin(\phi) \boxed{\quad} , \boxed{\quad} , \sin(\phi) \boxed{\quad}] \; ; .$$

b) (2 points) A second ellipsoid $x^2+(y+3)^2/9+z^2=1$ is parametrized as a surface of revolution

$$\vec{r}(\theta,y) = [\boxed{\quad} \cos(\theta), y, \boxed{\quad} \sin(\theta)] \; .$$

c) (2 points) The cone $y = x^2+z^2-3$ is parametrized as a graph above the xz-plane:

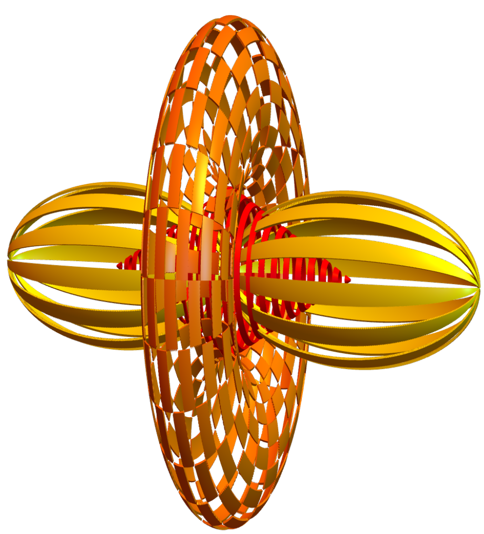
$$\vec{r}(x,z) = [\boxed{\quad} , x^2+z^2-3, \boxed{\quad}] \; .$$

d) (2 points) A second cone $y = -x^2-z^2+3$ is parametrized as a surface of revolution

$$\vec{r}(y,\theta) = [\boxed{\quad} , y, \boxed{\quad}]$$

e) (2 points) There is major **torus** given the points which are in distance 1 from the circle $\{y=0, x^2+z^2=4\}$. Complete the rest:

$$\vec{r}(\theta,\psi) = [(2+\cos(\psi)) \boxed{\quad} , \sin(\psi), (2+\cos(\psi)) \boxed{\quad}]$$



Solution:

a) $\boxed{\cos(\theta)}$ and $\boxed{3 + 3\cos(\phi)}$ and $\boxed{\sin(\theta)}$

b) $\boxed{\sqrt{1 - (y+3)^2/9}}$ and $\boxed{\sqrt{1 - (y+3)^2/9}}$

c) $\boxed{x}$ and $\boxed{z}$

d) $\boxed{\sqrt{3 - y}\cos(\theta)}$ and $\boxed{\sqrt{3 - y}\sin(\theta)}$.

e) $\boxed{\cos(\theta)}$ and $\boxed{\sin(\theta)}$