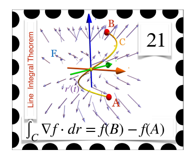
Triple Integrals

- $\iiint_E f(x,y,z)\ dzdydx = \iiint_E f\ dV$ triple integral
- $\int_a^b \int_c^d \int_u^v f(x,y,z)\ dzdydx$ integral over a cuboid, a rectangular box
- $\int_a^b [\iint_{G(z)} f(x,y,z)\ dA]dz$ hamburger cut, $G(z)$ =salad, cheese, tomato, beef slice
- $\iint_G [\int_{a(x,y)}^{b(x,y)} f(x,y,z)\ dz]dydx$ french fries cut, the inner integral are the fries
- $\iiint_E f(r,\theta,z)\ \boxed{r}\ dzdrd\theta$ integral in cylindrical coordinates
- $\iiint_E f(\rho,\theta,\phi)\ \boxed{\rho^2 \sin(\phi)}\ d\rho d\phi d\theta$ integral in spherical coordinates
- $\int_a^b \int_c^d \int_u^v f(x,y,z)\ dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\ dxdydz$ Fubini
- $\iiint_E \boxed{1}\ dzdydx = v(E)$ is the volume of solid E
- $\iiint_E \sigma(x,y,z)\ dV$ mass of solid E with density σ

Line Integrals

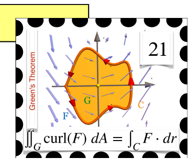
- $\vec{F}(x,y) = [P(x,y), Q(x,y)], \vec{F}(x,y,z) = [P(x,y,z), Q(x,y,z), R(x,y,z)]$ vector field.
- $\int_C \vec{F} \cdot \vec{dr} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ line integral
- $\vec{F}(x,y) = \nabla f(x,y)$ gradient field = potential field = conservative field

Fundamental theorem of line integrals



- FTL: $\vec{F}(x,y) = \nabla f(x,y)$, $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$
- Closed loop property $\int_C \vec{F}\ \vec{dr} = 0$ for all closed curves C
- Equivalent: closed loop property, path independence and gradient field
- Clairaut test: $\text{curl}(\vec{F}) \neq 0$ assures $\vec{F}$ is not a gradient field
- If simply connected domain: $\text{curl}(\vec{F}) = 0$ everywhere $\Rightarrow \vec{F} = \nabla f$

Green's Theorem



- $\vec{F}(x,y) = [P,Q]$, curl in two dimensions: $\text{curl}(\vec{F}) = Q_x - P_y$
- Green's theorem: C boundary of G, then $\int_C \vec{F} \cdot dr = \iint_G \text{curl}(\vec{F})\ dA$
- Take $\vec{F} = [-y, 0]$ or $\vec{F} = [0, x]$ to get area
- Green's theorem: compute difficult line integrals or difficult 2D integrals
- Orientation: the region G is to the left

Flux integrals

- $\vec{F}(x,y,z)$ vector field, $S = \vec{r}(G)$ parametrized surface
- $\vec{r}_u \times \vec{r}_v$ normal vector, $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ unit normal vector
- $\vec{r}_u \times \vec{r}_v\ dudv = \vec{dS} = \vec{n}\ dS$ normal surface element
- $\iint_S \vec{F} \cdot \vec{dS} = \iint_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dudv$ flux integral
- Orientation: the flux changes if the orientation of the surface is reversed

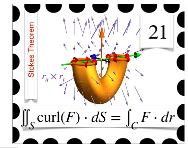
Stokes Theorem

$\vec{F}(x,y,z) = [P,Q,R]$, $\text{curl}([P,Q,R]) = [R_y - Q_z, P_z - R_x, Q_x - P_y] = \nabla \times \vec{F}$

Stokes's theorem: C boundary of surface S, then $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$

Stokes theorem: compute difficult flux integrals or difficult line integrals

Orientation: walk on boundary curve with surface "to the left"



Grad Curl Div

$\nabla = [\partial_x, \partial_y, \partial_z]$, $\vec{F} = \nabla f$, $\text{curl}(\vec{F}) = \nabla \times \vec{F}$, $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$

$\text{div}(\text{curl}(\vec{F})) = 0$ and $\text{curl}(\text{grad}(f)) = \vec{0}$

$\text{div}(\text{grad}(f)) = \Delta f$ Laplacian

incompressible = divergence free: $\text{div}(\vec{F}) = 0$ everywhere. Implies $\vec{F} = \text{curl}(\vec{G})$

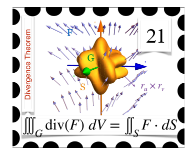
irrotational = $\text{curl}(\vec{F}) = 0$ everywhere. Implies $\vec{F} = \text{grad}(f)$

Divergence Theorem

$\text{div}([P,Q,R]) = P_x + Q_y + R_z = \nabla \cdot \vec{F}$

divergence theorem: solid E, boundary S then $\iint_S \vec{F} \cdot dS = \iiint_E \text{div}(\vec{F})\, dV$

divergence theorem: use to compute difficult flux or 3D integrals



Some topology

interior of region E: points in E for which small neighborhood is still in E

boundary of curve: the end points of the curve if they exist

boundary of surface S: points on S not in the interior: Example: rim of disc

boundary of solid E: part of the solid not in the interior of E

closed surface: a surface without boundary. Examples: sphere, torus

closed curve: a curve with no boundary. Examples: circle, knot.

Integration overview

Line integral: $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

Flux integral: $\iint_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dudv$

Theorems and Derivatives overview

1

1 $\xrightarrow{\text{grad}}$ 1

1 $\xrightarrow{\text{grad}}$ 2 $\xrightarrow{\text{curl}}$ 1

1 $\xrightarrow{\text{grad}}$ 3 $\xrightarrow{\text{curl}}$ 3 $\xrightarrow{\text{div}}$ 1

1

1 $\xrightarrow{\text{FTC}}$ 1

1 $\xrightarrow{\text{FTL}}$ 2 $\xrightarrow{\text{GREEN}}$ 1

1 $\xrightarrow{\text{FTL}}$ 3 $\xrightarrow{\text{STOKES}}$ 3 $\xrightarrow{\text{GAUSS}}$ 1

Oliver Knill, Math S 21a, last update: August 3, 2024

Integral Theorems Overview

INTEGRATION.

Line integral: $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$
Flux integral: $\int\int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v\; dudv$
Double integral: $\int\int_R f(x,y)\; dxdy$
Triple integral: $\int\int\int_E f(x,y,z)\; dxdydz$

Area $\int\int_R 1\; dA = \int\int_R 1\; dxdy$
Arc length $\int_a^b |\vec{r}\,'(t)|\; dt$
Surface area $\int\int |\vec{r}_u \times \vec{r}_v|\; dudv$
Volume $\int\int\int_E 1\; dxdydz$

DIFFERENTIATION.

Velocity: $\vec{r}\,'(t) = \frac{d}{dt}\vec{r}(t)$.
Partial derivative: $f_x(x,y,z)$.
Gradient: $\text{grad}(f) = [f_x, f_y, f_z]$
Curl in 2D: $\text{curl}([P,Q]) = Q_x - P_y$
Curl in 3D: $\text{curl}[P,Q,R] = [R_y - Q_z, P_z - R_x, Q_x - P_y]$
Div: $\text{div}(\vec{F}) = \text{div}[P,Q,R] = P_x + Q_y + R_z$.

IDENTITIES.

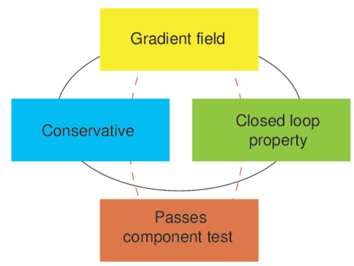
$\text{div}(\text{curl}(\vec{F})) = 0$
$\text{curl}(\text{grad}(f)) = \vec{0}$
$\text{div}(\text{grad}(f)) = \Delta f$.

JARGON.

$\text{div}(\vec{F}) = 0$ incompressible
$\text{curl}(\vec{F}) = \vec{0}$ irrotational

CONSERVATIVE FIELDS:

Gradient fields: $\vec{F} = \text{grad}(f)$.
Closed curve property: $\int_C \vec{F} \cdot \vec{dr} = 0$ for any closed curve.
Conservative: C_1, C_2 paths from A to B, then $\int_{C_1} \vec{F} \cdot \vec{dr} = \int_{C_2} \vec{F} \cdot \vec{dr}$.
Mixed derivative test: $\text{curl}(F) = 0$ if $\vec{F} = \nabla f$.
If $\vec{F}$ is irotational, then $\vec{\vec{F}} = \nabla f$ in simply connected R.



TOPOLOGY.
Interior of region D: points which have a neighborhood contained in D.
Boundary of curve: endpoints. **Boundary** of surface: curves. **Boundary** of solid: surfaces.
Simply connected: a closed curve in R can be deformed inside R to a point.
Closed curve Curve without boundary.
Closed surface surface without boundary.

LINE INTEGRAL THEOREM. If $C : \vec{r}(t) = [x(t), y(t), z(t)],\ t \in [a,b]$ is a curve and $f(x,y,z)$ is a function.

$$\int_C \nabla f \cdot \vec{dr} = f(\vec{r}(b)) - f(\vec{r}(a))$$

GREEN'S THEOREM. If R is a region with boundary C and $\vec{F} = [P,Q]$ is a vector field, then

$$\int\int_R \text{curl}(F)\; dxdy = \int_C \vec{F} \cdot \vec{dr}$$

STOKES THEOREM. If S is a surface with boundary C and $\vec{F}$ is a vector field, then

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot \vec{dr}$$

DIVERGENCE THEOREM. If S is the boundary of a solid E in space with boundary surface S and $\vec{F}$ is a vector field, then

$$\iiint_E \operatorname{div}(\vec{F})\, dV = \iint_S \vec{F} \cdot d\vec{S}$$

GENERAL STOKES. All theorems are of the form

$$\int_G dF = \int_{\delta G} F$$

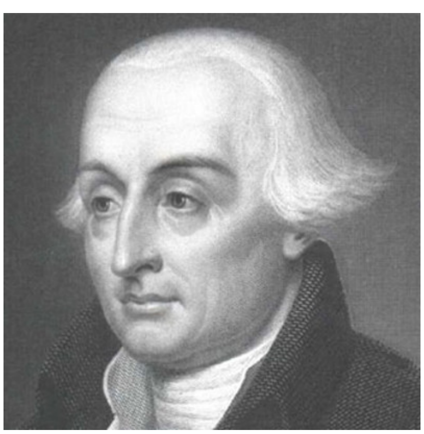
where dF is the derivative of F and δG is the oriented boundary of G.

George Gabriel Stokes

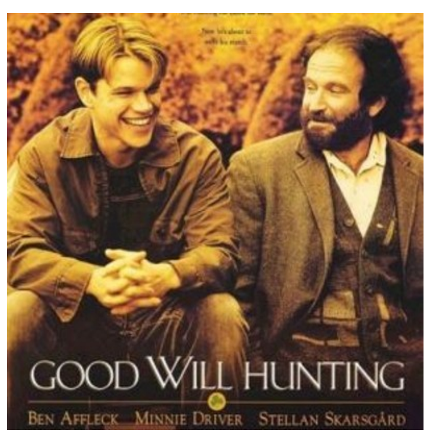
Mikhail Ostrogradsky

Carl Friedrich Gauss

André-Marie Ampère



Joseph Louis Lagrange



George Green

Last update: July 28, 2024, Harvard University Math21a, Oliver Knill