**MULTIVARIABLE CALCULUS**

MATH S-21A

# Unit 21: Green's theorem

## Lecture

**21.1.** Calculus in two dimensions knows **two** derivatives $\nabla$, curl, **two integrals** $\int_C \vec{F}\, dr$ and $\iint_R f(x,y)\, dA$ and **two** integral theorems: the **fundamental theorem of line integrals** $\int_C \nabla f \cdot dr = f(B) - f(A)$ as well as **Green's theorem** which we cover today. You might be used to think about two-dimensions as the xy-plane in three space, but we insist on not looking at an ambient space. [1]

**21.2.** The **curl** of a vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ is the scalar field $\text{curl}(\vec{F})(x,y) = \nabla \times \vec{F} = Q_x(x,y) - P_y(x,y)$. It measures the **vorticity** of the vector field at $(x,y)$. For example, for $\vec{F}(x,y) = [x^3 + y^2, y^3 + x^2 y]$, we have $\text{curl}(F)(x,y) = 2xy - 2y$.

**Theorem:** **Green's theorem** tells that if $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ is a vector field and $G$ is a region for which the boundary $C$ is a curve, parametrized so that $G$ is "to the left", then

$$\int_C \vec{F} \cdot \vec{dr} = \iint_G \text{curl}(\vec{F})\, dxdy\ .$$

**21.3.** Take a square $G = [x, x+h] \times [y, y+h]$ with small $h > 0$. The line integral of $\vec{F} = [P, Q]$ along the boundary $C$ is $\int_0^h P(x+t, y)dt + \int_0^h Q(x+h, y+t)\, dt - \int_0^h P(x+t, y+h)\, dt - \int_0^h Q(x, y+t)\, dt$. It measures the "circulation" at the position $(x,y)$. Because $Q(x+h,y) - Q(x,y) \sim Q_x(x,y)h$ and $P(x,y+h) - P(x,y) \sim P_y(x,y)h$, the line integral is $(Q_x - P_y)h^2$ which is $\int_0^h \int_0^h \text{curl}(\vec{F})\, dxdy$ up to an error of order $h^3$. A general region $G$ with area $|G|$ can be chopped into small squares of size $h$. We need about $|G|/h^2$ such squares. Summing up all the line integrals around the boundaries is the sum of the line integral along the boundary of $G$ because of the cancellations in the interior. On the boundary, it becomes a Riemann sum of the line integral along the boundary. The sum of the curls of the squares is a Riemann sum approximation of the double integral $\iint_G \text{curl}(\vec{F})\, dxdy$. Taking the limit $h \to 0$ gives Green's theorem.

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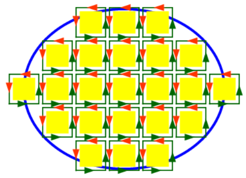
[1] We are "flat-landers" as in Edwin Abbot's novel (1884): "the plane" is the "universe".



**21.4.** **George Green** lived from 1793 to 1841. There is no single picture of him. Green was a physicist, a self-taught mathematician as well as a miller.

**21.5.** If $\vec{F}$ is a gradient field $\vec{F} = \nabla f$, then both sides of Green's theorem are zero: $\int_C \vec{F} \cdot \vec{dr}$ is zero by the fundamental theorem for line integrals. And $\iint_G \text{curl}(\vec{F}) \cdot dA$ is zero because $\text{curl}(\vec{F}) = \text{curl}(\text{grad}(f)) = 0$ everywhere on $G$.

**21.6.** If $\vec{F}(x,y) = \nabla f$ is a gradient field then the curl is zero because if $P(x,y) = f_x(x,y), Q(x,y) = f_y(x,y)$ and $\text{curl}(\vec{F}) = Q_x - P_y = f_{yx} - f_{xy} = 0$ by Clairaut. The field $\vec{F}(x,y) = [x+y, yx]$ for example is not a gradient field because $\text{curl}(\vec{F}) = y - 1$.

**21.7.** The already established **Clairaut identity**

$\text{curl}(\text{grad}(f)) = 0.$

**21.8.** can also be remembered by writing $\text{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\text{curl}(\nabla f) = \nabla \times \nabla f$. Use now that cross product of two identical vectors is 0. Working with $\nabla$ as a vector is called **nabla calculus**. It is useful as a mnemonic, to remember things.

**21.9.** It had been a consequence of the fundamental theorem of line integrals that:

If $\vec{F}$ is a gradient field then $\text{curl}(\vec{F}) = 0$ everywhere.

**21.10.** Is the converse true? The answer is "it depends". We need a definition:

**Definition:** A region $R$ is called **simply connected** if every closed loop in $R$ can be pulled together continuously within $R$ to a point inside $R$.

**21.11.** $R = \{x^2 + y^2 \leq 1\}$ is simply connected, $O = \{3 \leq x^2 + y^2 \leq 4\}$ is not.

If $\text{curl}(\vec{F}) = 0$ in a simply connected region $G$, then $\vec{F}$ is a gradient field.

**Proof.** Assume $R$ is inside a region $G$ and assume $\vec{F}$ is smooth in $G$. If $C$ in $G$ encloses a region $R$, then Green's theorem assures that for any gradient field $\vec{F}$, we have $\int_C \vec{F}\ \vec{dr} = 0$. So $\vec{F}$ has the closed loop property in $G$. This is equivalent to the fact that line integrals are path independent. In that case $\vec{F}$ is is gradient field:

define $f(x, y)$ by taking the line integral from an arbitrary point $O$ to $(x, y)$. There are non-simply connected region, where it is possible that $\text{curl}(\vec{F}) = 0$ does not imply that $\vec{F}$ is a gradient field.

### Examples

**21.12. Problem:** Find the line integral of $\vec{F}(x, y) = [x^2 - y^2, 2xy] = [P, Q]$ along the boundary of the rectangle $[0, 2] \times [0, 1]$. **Solution:** $\text{curl}(\vec{F}) = Q_x - P_y = 2y + 2y = 4y$. Apply Green to see $\int_C \vec{F}\ d\vec{r} = \int_0^2 \int_0^1 4y\ dydx = 2y^2|_0^1 x|_0^2 = 4$.

**21.13. Problem:** Find the area of the region enclosed by

$$\vec{r}(t) = [\frac{\sin^2(\pi t)}{t}, t^2 - 1]$$

for $-1 \leq t \leq 1$. To do so, use Greens theorem with the vector field $\vec{F} = [0, x]$. we do this example in class.

**21.14.** Green's theorem allows to express the coordinates of the **centroid** = center of mass

$$(\int \int_G x\ dA/A, \int \int_G y\ dA/A)$$

using line integrals. With $\vec{F} = [0, x^2/2]$ we have $\int \int_G x\ dA = \int_C \vec{F}\ d\vec{r}$.

**21.15.** An important application of Green is **area computation**: Take a vector field like $\vec{F}(x, y) = [P, Q] = [0, x]$ which has constant vorticity $\text{curl}(\vec{F})(x, y) = 1$. For $\vec{F}(x, y) = [0, x]$ , the right hand side in Green's theorem is the **area** $\text{Area}(G) = \int_C \vec{F}(\vec{r}(t))\ \vec{r}'(t)\ dt$.

**21.16.** Let $G$ be the region below the graph of a function $f(x)$ on $[a, b]$. The line integral around the boundary of $G$ is 0 from $(a, 0)$ to $(b, 0)$ because $\vec{F}(x, y) = [0, 0]$ there. The line integral is also zero from $(b, 0)$ to $(b, f(b))$ and $(a, f(a))$ to $(a, 0)$ because $P = 0$. The line integral along the curve $(t, f(t))$ is $-\int_a^b [-y(t), 0] \cdot [1, f'(t)]\ dt = \int_a^b f(t)\ dt$. Green's theorem confirms that this is the area of the region below the graph.

**21.17.** An engineering application of the theorem is the **planimeter**. This is a mechanical device designed to measure areas. We demonstrate it in class. Historically, it had been used in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leafs or estimate the wing size of insects, in agriculture it was used to measure the area of forests, in engineering to measure the size of profiles. There is a vector field $\vec{F}$ associated to the device which is obtained by placing a unit vector perpendicular to the arm. One can prove that $\vec{F}$ has constant curl 1. The planimeter calculates the line integral of $\vec{F}$ along a given curve. Green's theorem assures this is the area.

# Homework

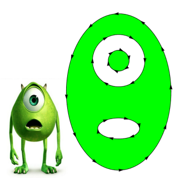
This homework is due on Tuesday, 8/6/2024.

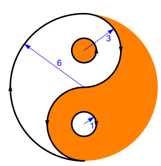
**Problem 21.1:** Given the Monster function $f(x, y) = 999 * x^{999} + 999 * x * y^{999}$, compute the line integral of $\vec{F}(x, y) = [25y + 6y^2, 12xy + 10y^9] + \nabla f$ along the boundary of the **Monster region** in the figure. There are 4 oriented boundary curves. The mouth (ellipse) of area 16, eyes (circles) of area 1 and 2 as well as a small ellipse (mouth) of area 3.

**Problem 21.2:** Find the area of the region bounded by the **hypocycloid** $\vec{r}(t) = [\cos^3(t), \sin^3(t)], 0 \leq t \leq 2\pi$.

**Problem 21.3:** Let $G$ be the region $x^{10} + y^{10} \leq 1$. Mathematica allows us to get the area as Area[ImplicitRegion[$\mathrm{x}^{10} + \mathrm{y}^{10}$ <= 1, {x, y}]] and tells, it is $A = 3.94293$ (which is using the Gamma function $4\Gamma(11/10)^2/\Gamma(6/5)$). What is the line integral of $\vec{F}(x, y) = [x^{800} + \sin(x) - 55y, y^{12} + \cos(y) + 4x]$ counter clockwise along the boundary of $G$ in terms of $A$.

**Problem 21.4:** Let $C$ be the boundary curve of the white Yang part of the Ying-Yang symbol in the disc of radius 6. You can see in the image that the curve $C$ has three parts, and that the orientation of each part is given. Compute $\int_C \vec{F} \cdot \vec{dr}$ for $\vec{F}(x, y) = [-y + \sin(e^x), 5x]$.






**Problem 21.5:** Use Green's Theorem to evaluate $\int_C [\sin(\sqrt{1 + x^7}) + 21y, 121x] \cdot d\vec{r}$, where $C$ is the boundary of the region $K(4)$. You see in the picture $K(0), K(1), K(2), K(3), K(4)$. The first $K(0)$ is an equilateral triangle of length 1. The second $K(1)$ is $K(0)$ with 3 equilateral triangles of length 1/3 added. $K(2)$ is $K(1)$ with $3 * 4^1$ equilateral triangles of length 1/9 added. **Remark.** In the limit $K = K(\infty)$, we get a **fractal** called the **Koch snowflake**.