MULTIVARIABLE CALCULUS

MATH S-21A

Unit 14: Lagrange

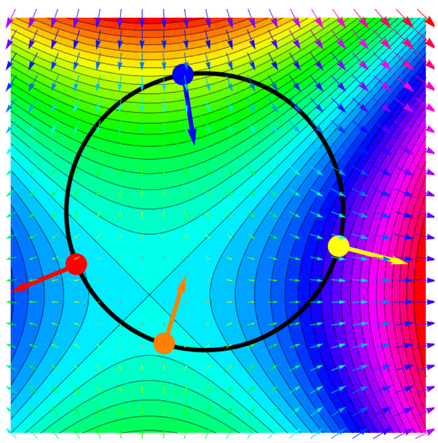
Lecture

14.1. When looking for maxima and minima of a function $f(x, y)$ in the presence of a **constraint** $g(x, y) = 0$, a necessary condition is that the gradients of f and g are parallel. The reason is that otherwise, we can move along the level curve $g = c$ and increase the value of f. Indeed, the directional derivative of f in the direction tangent to the level curve is zero if and only if the tangent vector to g is perpendicular to the gradient of f. Note that perpendicular can also include the case $\nabla g = [0, 0]$.

Definition: The system of equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = 0$ for the three unknowns x, y, λ are called the **Lagrange equations**. The variable λ is a **Lagrange multiplier**.

Theorem: A maximum or minimum of $f(x, y)$ on the curve $g(x, y) = c$ is either a solution of the Lagrange equations or then is a critical point of g.

Proof. The condition that ∇f is parallel to ∇g either means $\nabla f = \lambda \nabla g$ or $\nabla f = 0$ or $\nabla g = 0$. The case $\nabla f = 0$ can be included in the Lagrange equation case with $\lambda = 0$. The case $\nabla g = 0$ however needs to be added as a special case because in that case, $\nabla g = \lambda \nabla f$ works with $\lambda = 0$. QED.



14.2. In higher dimensions, the statement is exactly the same: extrema of $f(\vec{x})$ under the constraint $g(\vec{x}) = c$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g, g = c$ or then points, where $\nabla g = \vec{0}$. But we also can have more than one constraint:

Theorem: Extrema of $f(x, y, z)$ under the constraint $g(x, y, z) = c, h(x, y, z) = d$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$ or solutions to $\nabla g = 0, \nabla f(x, y, z) = \mu \nabla h, h = d$ or solutions to $\nabla h = 0, \nabla f = \lambda \nabla g, g = c$ or solutions to $\nabla g = \nabla h = 0$.

14.3. Remarks.
1) The conditions in the Lagrange theorem are equivalent to $\nabla f \times \nabla g = \vec{0}$ in dimensions 2 or 3 where we have a cross product.
2) With $g(x, y) = 0$, the Lagrange equations can also be written as $\nabla F(x, y, \lambda) = \vec{0}$, where $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$.
3) The two conditions in the theorem are equivalent to "$\nabla g = \lambda \nabla f$ or f has a critical point".
4) Constrained optimization problems work also in higher dimensions.
5) Can we avoid Lagrange? This is sometimes done in single variable calculus: in order to maximize xy under the constraint $2x + 2y = 4$ for example, we solve for y in the second equation and then solve the single variable problem $f(x, y(x))$. This needs to be done carefully. Boundaries must be considered. To extremize $f(x, y) = y$ on $x^2 + y^2 = 1$ for example we need to maximize $\sqrt{1 - x^2}$. We can differentiate to get the critical points but also have to look at the cases $x = 1$ and $x = -1$, where the actual minima and maxima occur. In general also, we can not do the substitution. To extremize $f(x, y) = x^2 + y^2$ with constraint $g(x, y) = x^4 + 3y^2 - 1 = 0$ for example, we solve $y^2 = (1 - x^4)/3$ and minimize $h(x) = f(x, y(x)) = x^2 + (1 - x^4)/3$. $h'(x) = 0$ gives $x = 0$. The find the maximum $(\pm 1, 0)$, we had to maximize $h(x)$ on $[-1, 1]$, which occurs at ± 1.
To extremize $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = p(x) + p(y) = 1$, where p is a complicated function in x which satisfies $p(0) = 0, p'(1) = 2$,the Lagrange equations $2x = \lambda p'(x), 2y = \lambda p'(y), p(x) + p(y) = 1$ can be solved with $x = 0, y = 1, \lambda = 1$. We can not solve $g(x, y) = 1$ however for y in an explicit way.
6) How do we determine whether a solution of the Lagrange equations is a maximum or minimum? Instead of introducing a second derivative test, we just make a list of critical points and pick the maximum and minimum. A second derivative test could be designed using second directional derivative in the direction of the tangent.
7) The Lagrange method also works with more constraints. The constraints $g = c, h = d$ define a curve in space. The gradient of f must now be in the plane spanned by the gradients of g and h because otherwise, we could move along the curve and increase f:

Examples

14.4. Minimize $f(x, y) = x^2 + 2y^2$ under the constraint $g(x, y) = x + y^2 = 1$. **Solution:** The Lagrange equations are $2x = \lambda, 4y = \lambda 2y$. If $y = 0$ then $x = 1$. If $y \neq 0$ we can divide the second equation by y and get $2x = \lambda, 4 = \lambda 2$ again showing $x = 1$. The point $x = 1, y = 0$ is the only solution.

14.5. Find the shortest distance from the origin to the curve $x^6 + 3y^2 = 1$. **Solution:** Minimize the function $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = x^6 + 3y^2 = 1$. The gradients are $\nabla f = [2x, 2y], \nabla g = [6x^5, 6y]$. The Lagrange equations $\nabla f = \lambda \nabla g$ lead to the system $2x = \lambda 6x^5, 2y = \lambda 6y, x^6 + 3y^2 - 1 = 0$. We get $\lambda = 1/3, x = x^5$, so that either $x = 0$ or 1 or -1. From the constraint equation $g = 1$, we obtain $y = \sqrt{(1 - x^6)/3}$. So, we have the solutions $(0, \pm\sqrt{1/3})$ and $(1, 0), (-1, 0)$. To see which is the minimum, just evaluate f on each of the points. $(0, \pm\sqrt{1/3})$ are the minima.

14.6. Which cylindrical soda cans of height h and radius r has minimal surface for fixed volume? **Solution:** The volume is $V(r, h) = h\pi r^2 = 1$. The surface area is $A(r, h) = 2\pi rh + 2\pi r^2$. With $x = h\pi, y = r$, you need to optimize $f(x, y) = 2xy + 2\pi y^2$ under the constrained $g(x, y) = xy^2 = 1$. Calculate $\nabla f(x, y) = (2y, 2x + 4\pi y), \nabla g(x, y) = (y^2, 2xy)$. The task is to solve $2y = \lambda y^2, 2x + 4\pi y = \lambda 2xy, xy^2 = 1$. The first equation gives $y\lambda = 2$. Putting that in the second one gives $2x + 4\pi y = 4x$ or $2\pi y = x$. The third equation finally reveals $2\pi y^3 = 1$ or $y = 1/(2\pi)^{1/3}, x = 2\pi(2\pi)^{1/3}$. This means $h = 0.54.., r = 2h = 1.08$. Remark: Other factors can influence the shape. For example, the can has to withstand a pressure up to 100 psi. A typical can of "Coca-Cola classic" with 3.7 volumes of CO_2 dissolve has at 75F an internal pressure of 55 psi, where PSI stands for pounds per square inch.

14.7. On the curve $g(x, y) = x^3 - y^2$ the function $f(x, y) = x$ obviously has a minimum $(0, 0)$. The Lagrange equations $\nabla f = \lambda \nabla g$ have no solutions. This is a case where the minimum is a solution to $\nabla g(x, y) = 0$.

14.8. Find the extrema of $f(x, y, z) = z$ on the sphere $g(x, y, z) = x^2 + y^2 + z^2 = 1$. Solution: compute the gradients $\nabla f(x, y, z) = (0, 0, 1)$, $\nabla g(x, y, z) = (2x, 2y, 2z)$ and solve $(0, 0, 1) = \nabla f = \lambda \nabla g = (2\lambda x, 2\lambda y, 2\lambda z), x^2 + y^2 + z^2 = 1$. The case $\lambda = 0$ is excluded by the third equation $1 = 2\lambda z$ so that the first two equations $2\lambda x = 0, 2\lambda y = 0$ give $x = 0, y = 0$. The 4'th equation gives $z = 1$ or $z = -1$. The minimum is the south pole $(0, 0, -1)$ the maximum the north pole $(0, 0, 1)$.

14.9. A dice shows k eyes with probability p_k with k in $\Omega = \{1, 2, 3, 4, 5, 6\}$. A probability distribution is a non-negative function p on Ω which sums up to 1. It can be written as a point $p = (p_1, p_2, p_3, p_4, p_5, p_6)$ with $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. The **entropy** of the probability vector $\vec{p}$ is defined as $S = -\sum_{i=1}^{6} p_i \log(p_i) = -p_1 \log(p_1) - p_2 \log(p_2) - \cdots - p_6 \log(p_6)$. Find the distribution p which maximizes entropy under the constrained $g(p) = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. **Solution:** $\nabla S = (-1 - \log(p_1), \dots, -1 - \log(p_n))$, $\nabla g = (1, \dots, 1)$. The Lagrange equations are $-1 - \log(p_i) = \lambda, p_1 + \cdots + p_6 = 1$, from which we get $p_i = e^{-(\lambda+1)}$. The last equation $1 = \sum_i \exp(-(\lambda + 1)) = 6 \exp(-(\lambda + 1))$ fixes $\lambda = -\log(1/6) - 1$ so that $p_k = 1/6$. The **uniform distribution** has maximal entropy meaning that it has the **least information content**. An unfair dice allows a gambler or casino to gain profit.

14.10. Assume that the probability that a chemical compound is in the state k is p_k and that the **energy** H of the state k is E_k. Nature tries to minimize the **free energy** $f(p_1, \dots, p_n) = S - H = -\sum_{k=1}^{n} p_k \log(p_k) - E_k p_k$ if the energies E_k are fixed. Using the constraint $\sum_k p_k = 1$, we have $\nabla f = [-1 - \log(p_1) - E_1, \dots, -1 - \log(p_n) - E_n]$, $\nabla g = [1, \dots, 1]$ and the Lagrange equation $\log(p_k) = -1 - \lambda - E_k$, so that $p_k = e^{-E_k} C$

for a constant C. The constraint $g(p) = p_1 + \cdots + p_n = 1$ gives $C(\sum_j e^{-E_j}) = 1$ so that $C = 1/(\sum_j e^{-E_j})$. We end up with $p_k = e^{-E_k}/\sum_{i=1}^n e^{-E_i}$. This **Gibbs distribution** p minimizes f.

15. Homework

This homework is due on Tuesday, 7/23/2024.

Problem 14.1: Find a cylindrical trash can which is open on the top has has the largest volume for fixed area π. If x is the radius and y is the height, we have to maximize $f(x, y) = \pi x^2 y$ under the constraint $g(x, y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.

Problem 14.2: A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi rh + \pi r^2$. We design a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes $f(h, r) = 3r^2 + 2rh$ under the constraint $g(h, r) = 2r^3 + 3r^2 h = 1$. Use these variables the Lagrange method to find a local minimum of f under the constraint $g = 1$.

Problem 14.3: Find the extrema of the same function

$$f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$$

you have seen in HW 13, but now on the entire disc $\{x^2 + y^2 \leq 4\}$ of radius 2. Besides the already found extrema **inside** the disk, now find also the extrema on the boundary.

Problem 14.4: We build a hot air balloon with a cuboid mesh of dimension x, y, z which together with the top and bottom fortifications uses wires of total length $g(x, y, z) = 6x + 6y + 4z = 32$. Find the balloon with maximal volume $f(x, y, z) = xyz$.

Problem 14.5: Which pyramid of height h over a square $[-a, a] \times [-a, a]$ with surface area is $4a\sqrt{h^2 + a^2} + 4a^2 = 4$ has maximal volume $V(h, a) = 4ha^2/3$? By using new variables (x, y) and multiplying V with a constant, we get to the equivalent problem to maximize $f(x, y) = yx^2$ over the constraint $g(x, y) = x\sqrt{y^2 + x^2} + x^2 = 1$. Use the later variables.