

# MULTIVARIABLE CALCULUS

MATH S-21A

## Unit 10: Linearization

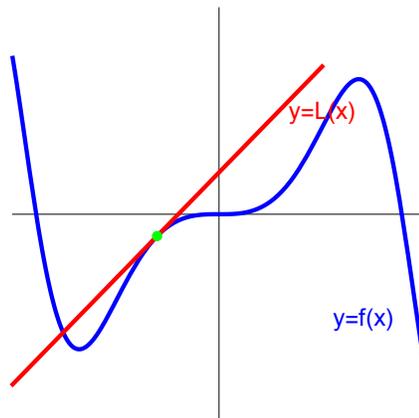
### LECTURE

**10.1.** Remember that in single variable calculus we have approximated functions near a point  $a$  with linear functions:

**Definition:** The **linear approximation** of  $f(x)$  at  $a$  is the affine function

$$L(x) = f(a) + f'(a)(x - a) .$$

**10.2.** If you remember **Taylor series**, this is the part of the series  $f(x) = \sum_{k=0}^{\infty} f^{(k)}(a)(x-a)^k/k!$ , where only the  $k = 0$  and  $k = 1$  term are considered. We think about the linear approximation  $L$  as a function and not as a graph because we also will look at linear approximations for functions of three variables, where we can not draw graphs.



**10.3.** The graph of the function  $L$  is close to the graph of  $f$  at  $a$ . What about higher dimensions?

**Definition:** The **linear approximation** of  $f(x, y)$  at  $(a, b)$  is the affine function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) .$$

The **linear approximation** of  $f(x, y, z)$  at  $(a, b, c)$  is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) .$$

**10.4.** Using the **gradient**

$$\nabla f(x, y) = [f_x, f_y], \quad \nabla f(x, y, z) = [f_x, f_y, f_z],$$

the linearization can be written more compactly as

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}).$$

**10.5.** How do we justify the linearization? If the second variable  $y = b$  is fixed, we have a one-dimensional situation, where the only variable is  $x$ . Now,  $f(x, b) = f(a, b) + f_x(a, b)(x - a)$  is the linear approximation. Similarly, if  $x = x_0$  is fixed  $y$  is the single variable, then  $f(x_0, y) = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$ . Knowing the linear approximations in both the  $x$  and  $y$  variables, we can get the general linear approximation by  $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ .

## EXAMPLES

**10.6.** What is the linear approximation of the function  $f(x, y) = \sin(\pi xy^2)$  at the point  $(1, 1)$ ? Answer: We have  $[f_x(x, y), f_y(x, y)] = [\pi y^2 \cos(\pi xy^2), 2xy\pi \cos(\pi xy^2)]$  which is at the point  $(1, 1)$  equal to  $\nabla f(1, 1) = [\pi \cos(\pi), 2\pi \cos(\pi)] = [-\pi, -2\pi]$ . The function is  $L(x, y) = 0 + (-\pi)(x - 1) - 2\pi(y - 1)$ .

**10.7.** Linearization can be used to estimate functions near a point. In the previous example,

$$f(1 + 0.01, 1 + 0.01) = -0.095$$

$$L(1 + 0.01, 1 + 0.01) = -\pi 0.01 - 2\pi 0.01 = -3\pi/100 = -0.0942.$$

**10.8.** Here is an example in three dimensions: find the linear approximation to  $f(x, y, z) = xy + yz + zx$  at the point  $(1, 1, 1)$ . Since  $f(1, 1, 1) = 3$ , and  $\nabla f(x, y, z) = (y + z, x + z, y + x)$ ,  $\nabla f(1, 1, 1) = [2, 2, 2]$ . we have  $L(x, y, z) = f(1, 1, 1) + [2, 2, 2] \cdot [x - 1, y - 1, z - 1] = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1) = 2x + 2y + 2z - 3$ .

**10.9.** Estimate  $f(0.01, 24.8, 1.02)$  for  $f(x, y, z) = e^x \sqrt{y}z$ .

**Solution:** take  $(x_0, y_0, z_0) = (0, 25, 1)$ , where  $f(x_0, y_0, z_0) = 5$ . The gradient is  $\nabla f(x, y, z) = (e^x \sqrt{y}z, e^x z / (2\sqrt{y}), e^x \sqrt{y})$ . At the point  $(x_0, y_0, z_0) = (0, 25, 1)$  the gradient is the vector  $(5, 1/10, 5)$ . The linear approximation is  $L(x, y, z) = f(x_0, y_0, z_0) + \nabla f(x_0, y_0, z_0)(x - x_0, y - y_0, z - z_0) = 5 + (5, 1/10, 5)(x - 0, y - 25, z - 1) = 5x + y/10 + 5z - 2.5$ . We can approximate  $f(0.01, 24.8, 1.02)$  by  $5 + (5, 1/10, 5) \cdot (0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$ . The actual value is  $f(0.01, 24.8, 1.02) = 5.1306$ , very close to the estimate.

**10.10.** Find the tangent line to the graph of the function  $g(x) = x^2$  at the point  $(2, 4)$ .

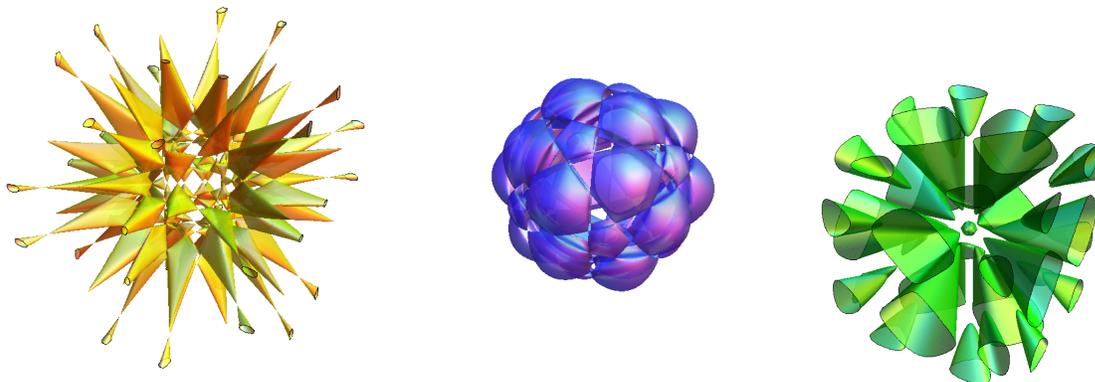
**Solution:** the level curve  $f(x, y) = y - x^2 = 0$  is the graph of a function  $g(x) = x^2$  and the tangent at a point  $(2, g(2)) = (2, 4)$  is obtained by computing the gradient  $[a, b] = \nabla f(2, 4) = [-g'(2), 1] = [-4, 1]$  and forming  $-4x + y = d$ , where  $d = -4 \cdot 2 + 1 \cdot 4 = -4$ . The answer is  $\boxed{-4x + y = -4}$  which is the line  $y = 4x - 4$  of slope 4.

**10.11.** The **Barth surface** is defined as the level surface  $f = 0$  of

$$f(x, y, z) = (3 + 5t)(-1 + x^2 + y^2 + z^2)^2(-2 + t + x^2 + y^2 + z^2)^2 + 8(x^2 - t^4 y^2)(-(t^4 x^2) + z^2)(y^2 - t^4 z^2)(x^4 - 2x^2 y^2 + y^4 - 2x^2 z^2 - 2y^2 z^2 + z^4),$$

where  $t = (\sqrt{5} + 1)/2$  is a constant called the **golden ratio**. If we replace  $t$  with  $1/t = (\sqrt{5} - 1)/2$  we see the surface to the middle. For  $t = 1$ , we see to the right the surface  $f(x, y, z) = 8$ . Find the tangent plane of the later surface at the point  $(1, 1, 0)$ .

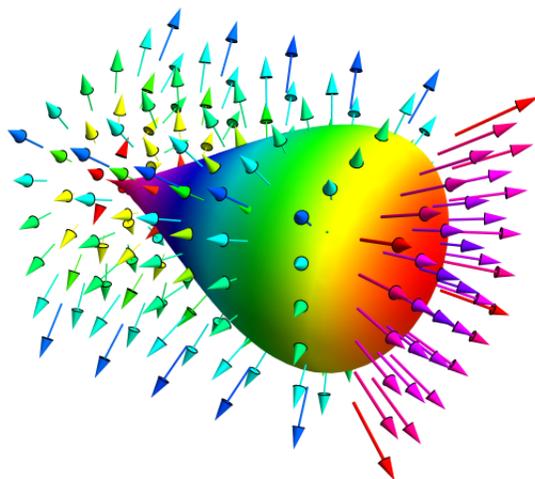
**Answer:** We have  $\nabla f(1, 1, 0) = [64, 64, 0]$ . The surface is  $x + y = d$  for some constant  $d$ . By plugging in  $(1, 1, 0)$  we see that  $x + y = 2$ .



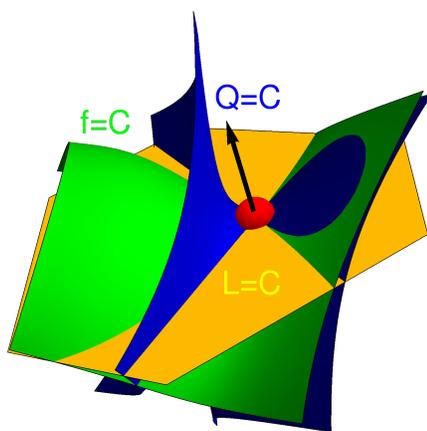
The quartic surface

$$f(x, y, z) = x^4 - x^3 + y^2 + z^2 = 0$$

is called the **piriform**. What is the equation for the tangent plane at the point  $P = (2, 2, 2)$  of this pair shaped surface? We get  $[a, b, c] = [20, 4, 4]$  and so the equation of the plane  $20x + 4y + 4z = 56$ , where we have obtained the constant to the right by plugging in the point  $(x, y, z) = (2, 2, 2)$ .



**10.12.** Linearization is just the first step for more accurate approximations. One could do **quadratic approximations** for example. In one dimension, one has  $Q(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x-a)^2}{2!}$ . In two dimensions, this becomes  $Q(x, y) = L(x, y) + H(a, b)[x - a, y - b] \cdot [x - a, y - b]/2$ , where  $H$  is the **Hessian matrix**  $H(a, b) = \begin{bmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{bmatrix}$ . We will see this matrix next week, when we maximize or minimize functions.



### HOMEWORK

This homework is due on Tuesday, 7/16/2024 in class.

**Problem 10.1:** Estimate  $10000000000000^{1/5}$  using linear approximation of  $f(x) = x^{1/5}$  near  $x_0 = 400^5$ .

**Problem 10.2:** Given  $f(x, y) = \frac{6yx}{\pi} - \cos(x)$ . Estimate  $f(\pi + 0.01, \pi - 0.03)$  using linearization

**Problem 10.3:** Estimate  $f(0.003, 0.9999)$  for  $f(x, y) = \cos(\pi y) + \sin(x + \pi y)$  using linearization.

**Problem 10.4:** Find the linear approximation  $L(x, y)$  of the function

$$f(x, y) = \sqrt{10 - x^2 - 5y^2}$$

at  $(2, 1)$  and use it to estimate  $f(1.95, 1.04)$ .

**Problem 10.5:** Estimate  $(99^3 * 101^2)$  by linearising the function  $f(x, y) = x^3y^2$  at  $(100, 100)$ . What is the difference between  $L(100, 100)$  and  $f(100, 100)$ ?