

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

|        |  |     |
|--------|--|-----|
| 1      |  | 20  |
| 2      |  | 10  |
| 3      |  | 10  |
| 4      |  | 10  |
| 5      |  | 10  |
| 6      |  | 10  |
| 7      |  | 10  |
| 8      |  | 10  |
| 9      |  | 10  |
| 10     |  | 10  |
| 11     |  | 10  |
| 12     |  | 10  |
| 13     |  | 10  |
| Total: |  | 140 |

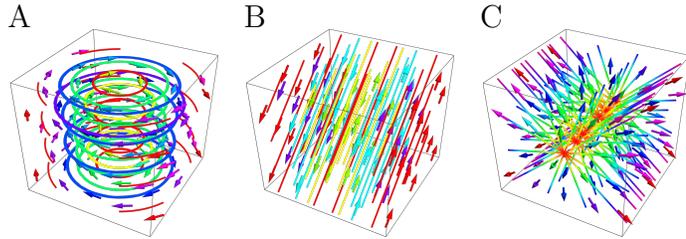
Problem 1) (20 points) No justifications are needed.

- 1)  T  F All vectors  $\vec{v}$  in space define a direction  $\vec{v}/|\vec{v}|$ .
- 2)  T  F The vectors  $[1, 3, 1]$  and  $[1, -1, 2]$  are perpendicular.
- 3)  T  F The curvature of the curve  $\vec{r}(t) = [\sin(\sin(t)), \cos(\sin(t)), 3]$  is 1 at  $t = 0$ .
- 4)  T  F Clairaut's theorem assures that  $f_{xyx} = f_{yxx}$  for every smooth function  $f(x, y)$ .
- 5)  T  F Maxwell took an exam in which he had to prove Stokes theorem.
- 6)  T  F The boundary of the ball  $x^2 + y^2 + z^2 \leq 1$  is the sphere  $x^2 + y^2 + z^2 = 1$ .
- 7)  T  F The boundary of the sphere  $x^2 + y^2 + z^2 = 1$  is the circle  $x^2 + y^2 = 1, z = 0$ .
- 8)  T  F The divergence of the gradient of the divergence of a field is always zero. In other words:  $\text{div}(\text{grad}(\text{div}(\vec{F}))) = 0$ .
- 9)  T  F We can use the divergence theorem to compute the volume of a solid.
- 10)  T  F The line  $\vec{r}(t) = [t, 2t, -3t]$  hits the plane  $x + 2y - 3z = 10$  orthogonally.
- 11)  T  F The quadratic surface  $x^2 + y^2 - z^2 = -1$  is a one-sheeted hyperboloid.
- 12)  T  F For non-zero vectors  $\vec{u}, \vec{v}$ , the relation  $|\vec{u} \times \vec{v}| = |\vec{u} \cdot \vec{v}|$  implies that  $\vec{u}, \vec{v}$  meet in a  $\pi/4$  or  $3\pi/4$  angle.
- 13)  T  F  $\int_0^{\pi/2} \int_0^3 r^5 d\theta dr = \int_0^3 \int_0^3 (x^2 + y^2)^2 dx dy$ .
- 14)  T  F If  $\vec{F} = [P, Q]$  satisfies  $\text{curl}(\vec{F})(x, y) = 0$  and  $\text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y) = 0$  for all points  $(x, y)$ , then  $\vec{F}$  is a constant field.
- 15)  T  F The acceleration  $\vec{r}''(t) = [x(t), y(t), z(t)]$ , the velocity vector  $\vec{r}'(t)$  and  $\vec{r}'(t) \times \vec{r}''(t)$  are pairwise perpendicular.
- 16)  T  F The curvature of the curve  $\vec{r}(t) = [\sin(2t), 0, \cos(2t)]$  is equal to the curvature of the curve  $\vec{s}(t) = [0, \cos(3t), \sin(3t)]$ .
- 17)  T  F The space curve  $\vec{r}(t) = [t \sin(t), t \cos(t), t^2]$  for  $t \in [0, 10\pi]$  is located on a paraboloid.
- 18)  T  F If a smooth function  $f(x, y)$  has a global maximum, then this maximum is a critical point.
- 19)  T  F If  $L(x, y)$  is the linearization of  $f(x, y)$  and  $\vec{s}(t)$  is the line tangent to the curve  $\vec{r}(t)$  at  $t_0$ . Then  $d/dt L(\vec{s}(t)) = d/dt f(\vec{r}(t))$  at the time  $t = t_0$ .
- 20)  T  F The flux of an incompressible vector field is zero through any closed surface  $S$  in space.

Problem 2) (10 points) No justifications are necessary.

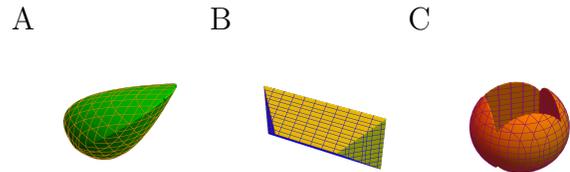
a) (2 points) The figures display vector fields in space. There is an exact match.

| Field                  | A-C |
|------------------------|-----|
| $\vec{F} = [0, x, x]$  |     |
| $\vec{F} = [y, -x, 0]$ |     |
| $\vec{F} = [x, 0, z]$  |     |



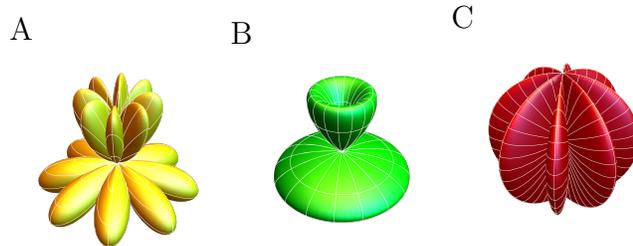
b) (2 points) Match the solids. There is an exact match.

| Solid                                  | A-C |
|--|-----|
| $x^2 + y^2 + z^2 < 5,  x  +  y  > 2$   |     |
| $ x  +  y  < 2,  y  +  z  < 1$         |     |
| $x^2 - y^2 < 1 + z, y^2 + z^2 < 1 + x$ |     |



c) (2 points) Match the surfaces

| Surface                              | A-C |
|--------------------------------------|-----|
| $\rho = \sin(4\phi)$                 |     |
| $\rho = \sin(4\theta) + \sin(4\phi)$ |     |
| $\rho = \sin(4\theta)$               |     |

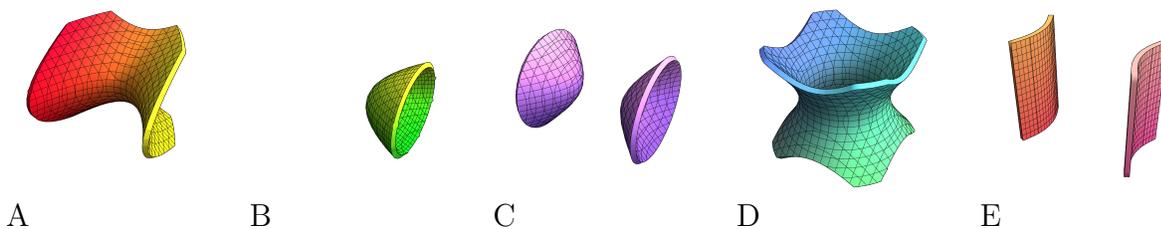


d) (2 points) What is the name and formula of the partial differential equation is used to model waves reaching a beach?

|       |           |
|-------|-----------|
| Name: | Equation: |
|       |           |

e) (2 points) Just to double check that you still have all your marbles quadrics in your head:

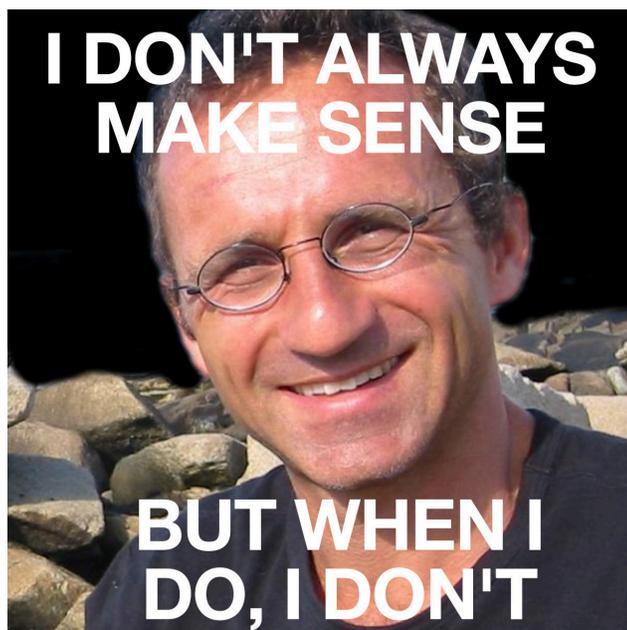
|                                  | Enter a letter from A-E in each of the three cases |
|----------------------------------|--|
| Pick the hyperbolic paraboloid   |  |
| Pick the two-sheeted hyperboloid |  |
| Pick the cylindrical hyperboloid |  |



Problem 3) (10 points) No justifications necessary

Which of the following expressions make sense? Each question is one point.

| Expression   | Makes sense | Does not make sense |
|--|-------------|---------------------|
| The double integral $\iint x^2 dx dy$                                      |             |                     |
| $\iiint \operatorname{div}(\operatorname{grad}(x^2 + y^2 + z^2)) dx dy dz$ |             |                     |
| The vector field $\operatorname{grad}(\operatorname{curl}(\vec{F}))$       |             |                     |
| The distance between two lines   |             |                     |
| The distance between two vector fields                                     |             |                     |
| The directional derivative $D_{\vec{v}}\vec{F}$                            |             |                     |
| The flow line of a parametrized surface                                    |             |                     |
| The traces of an implicit surface  |             |                     |
| The flux of a scalar function $f$ through a sphere                         |             |                     |
| The arc length of a closed flow line of a vector field                     |             |                     |



Problem 4) (10 points)

**Noah Lyles** recently clocked 19.31 seconds for the 200 meters sprint in Oregon. This is faster than the 19.32 of Michael Johnson in 1996. The first part of the sprint are run on a semi-circle or radius 36.50 meters. We look at a fellow runner who follows a circle of radius 40 meters, meaning

$$\vec{r}(t) = [40 \cos(at), 40 \sin(at), 0] .$$

Assuming this runner runs the 200 meters in 20 seconds, his speed is  $|\vec{r}'(t)| = a40 = 10$  in meters per second, giving him an angular velocity of  $a = 10/40 = 1/4$ . The upshot of this introduction is that we have a precise curve tracing the runner:

$$\vec{r}(t) = [40 \cos(\frac{t}{4}), 40 \sin(\frac{t}{4}), 0] .$$

- (2 points) What is the curvature of the path at  $t = \pi$ ?
- (2 points) Compute the vectors  $\vec{T}, \vec{N}, \vec{B}$  at  $t = \pi$ .
- (2 points) Use the arc length formula to get the arc length for  $t \in [0, \pi/2]$ .
- (2 points) What is the magnitude of the acceleration  $|r''(t)|$  at  $t = \pi$ ?
- (2 points) Finally compute the jerk  $\vec{r}'''(t)$  at  $t = \pi$ .



Problem 5) (10 points)

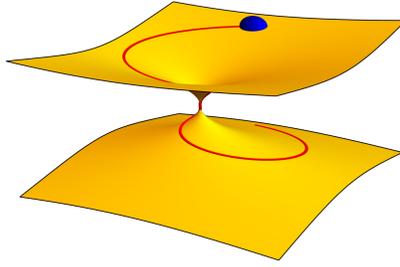
The curve

$$\vec{r}(t) = [1 + t^2 \sin(\pi t), t^2 \cos(\pi t), t]$$

is located on the surface  $S$  given by the implicit equation

$$(x - 1)^2 + y^2 = z^4 .$$

- (5 points) Find the tangent plane  $ax + by + cz = d$  to the surface at the point  $P = (1, 4, 2)$ .
- (5 points) Parametrize the tangent line through the point  $P$  containing the velocity vector of  $\vec{r}(t)$  at  $\vec{r}(2) = (1, 4, 2)$ .



Problem 6) (10 points)

a) (3 points) Find the **surface area** of the parallelogram with vertices

$$A = (0, 0, 0), B = (1, 1, 0), C = (2, 3, 3), D = (1, 2, 3)$$

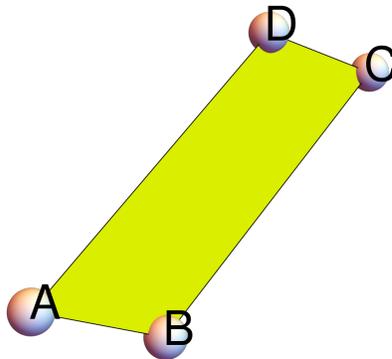
without integrating. Just compute it using a suitable cross product.

b) (3 points) Now find the surface area by using the parametrization

$$\vec{r}(t, s) = [t + s, t + 2s, 3s], 0 \leq t \leq 1, 0 \leq s \leq 1$$

and writing down an integral. We need in this part b) to see a double integral, and not just the answer.

c) (4 points) What is the distance from  $C$  to the line through  $A, B$ ?



Problem 7) (10 points)

Just after the second midterm, we started to compute **triple integrals**. Let us remember this wonderful moment because the hours after an exam are the sweetest in any school. We will also enjoy the afternoon after this final. What is nicer than slurping on an **iced vanilla caramel latte**, while Oliver has to grade exams on a hot day of 37 degrees celsius. Compute the following 3D integral:

$$\int_0^{\pi/2} \int_y^{\pi/2} \int_0^x \frac{\cos(2z)}{2x} dz dx dy .$$

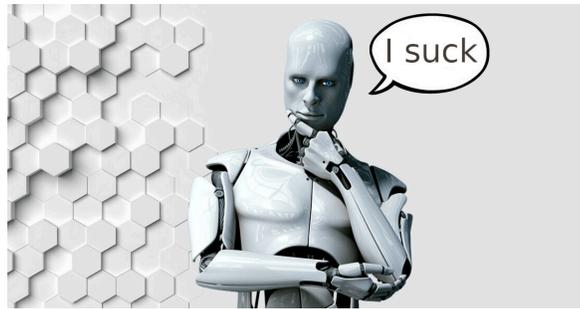


Problem 8) (10 points)

In an other attempt to derail a computer algebra system we give it the problem to classify the critical points of the infamous **1001-function**. Oliver has mentioned this example already in class. He likes to collect problems in which computers **suck** and get **stuck** but humans **succeed** and **lead**. He also likes to **rhyme** from time to **time**.

$$f(x, y) = x - \frac{x^{1001}}{1001} + y - \frac{y^{1001}}{1001}.$$

We also want to know whether the function  $f$  has a global max or min.

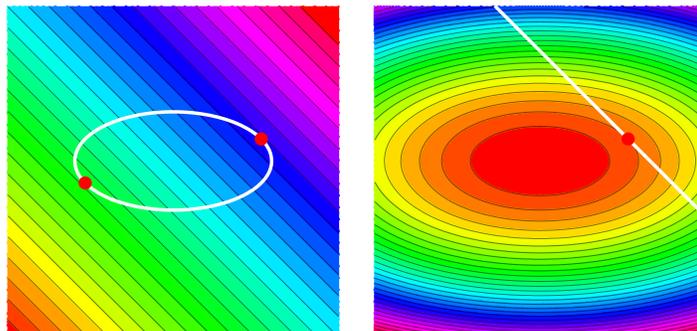


Problem 9) (10 points)

We would naively expect that extremizing  $f$  under the constraint  $g$  is equivalent to extremizing  $g$  under constraing when  $f$  is constant. This is not the case as we can see here:

a) (5 points) Find the minimum of  $f(x, y) = 4 + x^2 - 8y + 4y^2$  under the constraint  $g(x, y) = x + y = 11$  using Lagrange's method. You see only one solution.

b) (5 points) Now find the maximum and minimum of  $f(x, y) = x + y$  under the constraint  $g(x, y) = 4 + x^2 - 8y + 4y^2 = 80$  again using the method of Lagrange multipliers. You see two solutions of the Lagrange equations. Identify which is the maximum and which is the minimum.



Problem 10) (10 points)

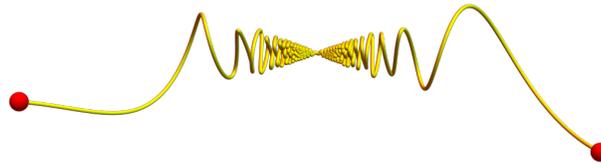
Find the **line integral**  $\int_C \vec{F} \cdot d\vec{r}$  for

$$\vec{F}(x, y, z) = [x^2 + yz, y^2 + xz, z^2 + xy]$$

along the **devil curve**

$$\vec{r}(t) = [3t, t \sin(\pi t) \sin(t + \frac{1}{0.001 + t^2}), t \sin(\pi t) \cos(t + \frac{1}{0.001 + t^2})]$$

with  $-1 \leq t \leq 1$ . Note that we are close to a curve with discontinuities it was smoothed out. The curve looks evil, but it is not!



Problem 11) (10 points)

What is the flux of the vector field

$$\vec{F} = [5x + yz, 3z - xz, xy + x + y + z]$$

through the surface  $S$  bounding the solid  $E$  seen in the picture? The solid  $E$  was obtained by taking a **hollow sphere**

$$9 < x^2 + y^2 + z^2 < 16$$

and slicing it into 20 equal pieces, then taking out every second, to half its volume. The surface is oriented outwards.



Problem 12) (10 points)

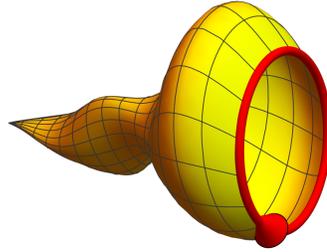
Let

$$\vec{F} = [z, y^2 e^x \cos(\cos(xyz)), -x]$$

be a vector field and  $S$  be the surface

$$\vec{r}(s, t) = [2s \cos(t)(1 - \frac{\sin(4\pi s)}{2}) + s \sin(2\pi s), 8s - 4, 2s \sin(t)(1 + \frac{\sin(3\pi s)}{2})]$$

seen in the picture, where  $t \in [0, 2\pi]$  and  $s \in [0, 1]$ . The surface orientation is defined by the parametrization. We assure you that it is outwards. There is one boundary curve, obtained by putting  $s = 1$ . What is the flux of the curl of  $\vec{F}$  through  $S$ ?



Problem 13) (10 points)

We still remember the wonderful Toblerone chocolate eaten during the review. What is the area of the **Toblerone region** enclosed by the curve

$$\vec{r}(t) = [t^3 - t, 1 - t^4]$$

with  $t \in [-1, 1]$ ?

