8/8/2024 PRACTICE 1 **Maths 21a, O.Knill, Summer 2024**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-2 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are needed.

1) T F The ellipse $4x^2 + 9y^2 = 1$ is parametrized by $\vec{r}(t) = [2\cos(t), 3\sin(t)]$.

2) T F The vector fields $\vec{F} = [x, y+z, z]$ and $\vec{G} = [-y, x, -x]$ are perpendicular at every point.

3) T F The curvature of the curve $\vec{r}(t) = [7+2\cos(t), 3, 7+2\sin(t)]$ is 2 everywhere.

4) T F The partial differential equation $f_{xxy} + f_{yxx} - 2f_{xyx} = 0$ is solved by every smooth function $f(x, y, z)$.

5) T F Stokes took an exam in which he had to prove the Maxwell's equations.

6) T F The boundary of the ball $x^2 + y^2 + z^2 \leq 1$ is the equator $x^2 + y^2 = 0, z = 0$

7) T F The boundary of the disk $x^2 + y^2 \leq 1$ consists of the circle $x^2 + y^2 = 1$.

8) T F The gradient of the divergence of the gradient of a vector field is always zero. In other words: $\text{grad}(\text{div}(\text{grad}(f))) = \vec{0}$.

9) T F We can use the Stokes theorem to compute the volume of a solid.

10) T F The circle $[\cos(t), \sin(t), 0]$ intersects with the circle $[0, \cos(t), \sin(t)]$ in at least one point.

11) T F The quadratic surface $-x^2 + y^2 - z^2 = -4$ is a two-sheeted hyperboloid.

12) T F If $\vec{r}(u, v)$ parametrizes a surface, then $\vec{r}_u$ and $\vec{r}_v$ are perpendicular.

13) T F The curl $\nabla \times \vec{F}$ of $\vec{F} = [P, Q, R]$ is a vector field that is always perpendicular to $\vec{F}$.

14) T F If $\vec{r}(t)$ parametrizes a curve, then the third derivative $\vec{r}'''(t)$ is called the jerk.

15) T F For any vector field $\vec{F}$ we have $\text{div}(\text{grad}(\text{div}(\vec{F}))) = 0$ at every point.

16) T F The triple scalar product is always positive.

17) T F If $\vec{v} = \nabla f / |\nabla f|$ at a non-critical point, then the directional derivative of f in the direction of $\vec{v}$ is positive.

18) T F The vector projection $\vec{Proj}_{\vec{v}}(\vec{w})$ of $\vec{w}$ onto $\vec{v}$ is a vector which has length smaller or equal than $|\vec{w}|$

19) T F Let $f(x, y)$ be a function with critical point $(0, 0)$ such that $f_{xx}(0, 0) = 1$, This assures that we have now either a local max or local min at $(0, 0)$.

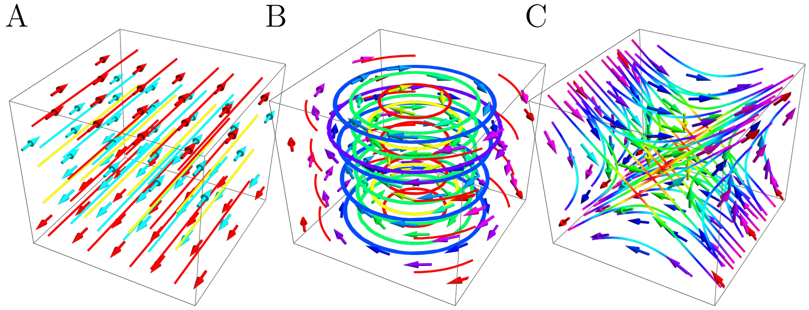
20) T F The curvature of a curve is $\frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) The following figures display vector fields in space. There is an exact match.

Field	A-C
$\vec{F} = [z, 0, x]$	
$\vec{F} = [0, z, 0]$	
$\vec{F} = [y, -x, 0]$	

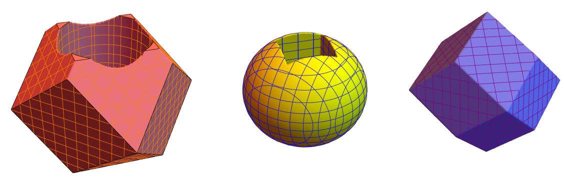
A B C



b) (2 points) Match the solids. There is an exact match.

Solid	A-C
$\|x\| + \|y\| < 1, \|y\| + \|z\| < 1, \|x\| + \|z\| < 1$	
$x^2 + y^2 + z^2 < 2, \|x\| + \|y\| > 1$	
$\|x\| + \|y\| + \|z\| < 2, x^2 + y^2 > 1$	

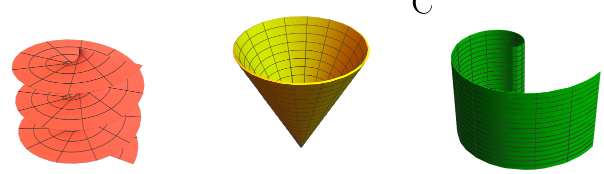
A B C



c) (2 points) Match the surfaces given either in cylindrical coordinates (r, θ, z) or parametrized:

Surface	A-C
$r = z$	
$r = \theta$	
$\vec{r}(r, \theta) = [r\cos(\theta), r\sin(\theta), \theta]$	

A B C

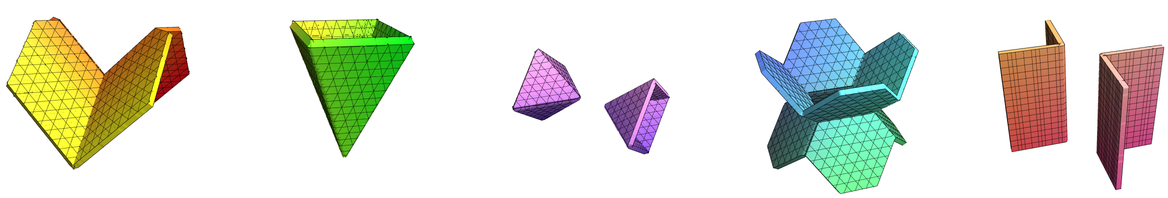


d) (2 points) What is the name and the formula of the basic **partial differential equation** which has the property that the solutions all converge to zero for large t.

Name:	Equation:

e) (2 points) In the first homework, we looked at **gems**, quadrics given the **taxi metric**. The sphere for example is $|x| + |y| + |z| = 1$ the paraboloid is $|z| = |x| + |y|$. They look like the round quadrics but have **stealth bomber** quality.

	Enter a letter from A-E in each of the three cases
Pick the one-sheeted hyperboloid	
Pick the hyperbolic paraboloid	
Pick the two-sheeted hyperboloid	



A B C D E

Problem 3) (10 points)

You are an **AI researcher** evaluating the abilities of an AI. We actually asked these questions to Chat GPT and reported the final conclusion. The AI of course gave a detailed solution.

a) (2 points) You: "What is the curvature of a circle of radius 5." The AI answers: $\boxed{1/5}$.

Is this correct?	Verify or state a general fact

b) (2 points) You: "Give me a parametrization of a torus"

The AI answers:

$$\begin{array}{rcl} x & = & (R + r\cos(\psi))\cos(\theta \\ y & = & (R + r\cos(\psi))\sin(\theta) \\ z & = & r\sin(\psi) \end{array}$$

Is this correct?	What is the meaning of r?

c) (2 points) You: "Compute the distance of the point (1,2,3) to the line r(t) = (t,t,t)." The AI gives the answer $\boxed{\sqrt{6}/3}$.

Is this correct?	How do you verify? Either give the distance formula or the idea.

d) (2 points) You: "Give me the distance of the line r(t) = (t,0,0) and the line r(t) = (t,1,0). The AI: $\boxed{1}$.

Is this correct?	Give your computation or an argument.

e) (2 points) You: "Is it true that if curl(F)=0 everywhere, then F is a gradient field?" AI:

"Yes, if a vector field F has a curl that is everywhere zero and it is defined on a simply-connected domain, then F is a conservative vector field, which means it can be represented as the gradient of a scalar potential function. In other words, F is a gradient field."

Is this correct?	Elaborate by telling which theorem or theorems can show this

Problem 4) (10 points)

A rocket at the **1. August celebration** at the rheinfalls in Switzerland experiences an acceleration

$$\vec{r}\,''(t) = \begin{bmatrix} 1 \\ 3 \\ e^t \end{bmatrix} .$$

Find the position $\vec{r}(t)$ of the rocket assuming that $\vec{r}(0) = [0, 0, 0]$ and $\vec{r}\,'(0) = [1, 1, 1]$.

Picture: From the Schaffhauser Nachrichten (newspaper in Switzerland) printed 1. August 2023

Problem 5) (10 points)

a) (4 points) Find the **tangent plane** to the surface $f(x, y, z) = x^2 y^4 z^6 = 1$ at the point $(1, 1, 1)$.

b) (3 points) **Estimate** $f(1.01, 1.001, 1.0001)$ using the method of linearization.

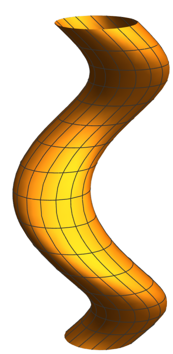
c) (3 points) Find the **directional derivative** $D_{\vec{v}} f$ at $(1, 1, 1)$ in the direction $\vec{v} = [1, 1, 1]/\sqrt{3}$.

Problem 6) (10 points)

a) (5 points) Write down the double integral of the **surface area** A of the surface

$$\vec{r}(t, s) = \begin{bmatrix} \cos(s) + \sin(3t) \\ \sin(s) + \cos(3t) \\ 3t \end{bmatrix}$$

with $0 \leq s \leq 2\pi$ and $0 \leq t \leq \pi$. There is no need to evaluate the final integral.



b) (5 points) Compute the **arc length** L of the center curve of the surface in a)

$$\vec{r}(t) = \begin{bmatrix} \sin(3t) \\ \cos(3t) \\ 3t \end{bmatrix}$$

with $0 \leq t \leq \pi$. Here we want a numerical answer! P.S. The **Pappus centroid theorem** will assure that the numerical answer of a) is $A = 2\pi L$.

Problem 7) (10 points)

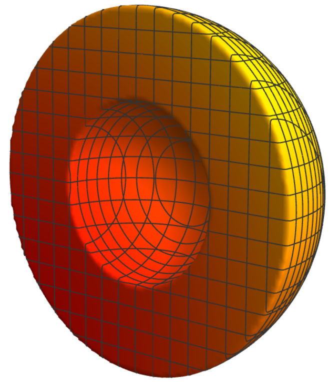
The **triple integral**

$$\iiint_E \frac{e^{-x^2-y^2-z^2}}{\sqrt{x^2+y^2+z^2}}\, dV$$

expresses the mass of the solid

$$E = \{1 \leq x^2 + y^2 + z^2 \leq 4, y > 0\}$$

with the given density. Find the mass.



Problem 8) (10 points)

Find the maxima and minima of

$$f(x, y) = 3xy + x^3 + y^3$$

using the second derivative test. As a neat student, you fill out a beautiful organized table, list all the points, and the ingredients which are needed to decide the nature of the critical points.

Point			Nature of the critical point

Problem 9) (10 points)

Use the Lagrange method to find the local maxima and minima of

$$f(x, y) = x^4 + y^4$$

under the constraint

$$g(x, y) = x^2 + y^2 = 2 \; .$$

There will be 8 critical points. Hunt them all down!

Oliver asked an AI to oil-paint his **medieval fantasy** of a "deer hunting down a human."

Problem 10) (10 points)

Compute the total line integral

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

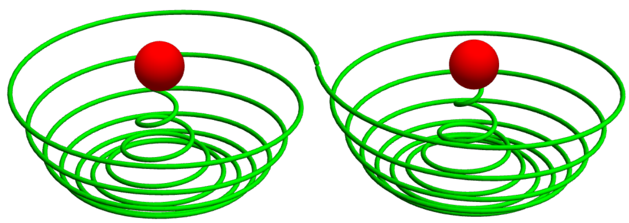
of

$$\vec{F}(x,y,z) = \begin{bmatrix} yze^{xyz} + 2x \\ xze^{xyz} + 2y \\ xye^{xyz} + 2z \end{bmatrix}$$

along the curves C_1, C_2 parametrized both by $0 \leq t \leq 2\pi$ and

$$\vec{r}_1(t) = \begin{bmatrix} t\cos(10t) \\ t\sin(10t) \\ t(\pi - t/2) \end{bmatrix}, \vec{r}_2(t) = \begin{bmatrix} 2\pi + t\cos(10t) \\ -t\sin(10t) \\ t(\pi - t/2) \end{bmatrix}.$$

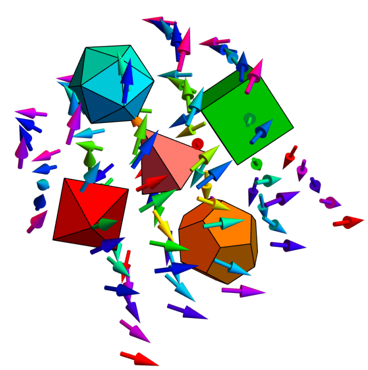
The path C_1 starts at the point $A = (0,0,0)$ and ends at the point $P = (2\pi, 0, 0)$. The second curve C_2 starts at the same point $P = (2\pi, 0, 0)$ and ends at the point $B = (4\pi, 0, 0)$.



Problem 11) (10 points)

Express the flux $\iint_S \vec{F} \cdot d\vec{S}$ of the vector field

$$\vec{F} = \begin{bmatrix} -100y \\ 100x \\ 10z \end{bmatrix}$$



through the surfaces bounding the disjoint five Platonic solids **Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron**. Their volumes are

$$0.117851, 1, 0.471405, 7.66312, 2.18169$$

and add up to $\boxed{11.4341}$. All solids have surface boundaries oriented outwards.
P.S. We have computed the volumes with the following Mathematica line

```
Vol[X_]:=N[Volume[ConvexHullMesh[PolyhedronData[X][[1 ,1]]]]];
P=PolyhedronData["Platonic"]; V=Map[Vol,P]; Total[V];
```

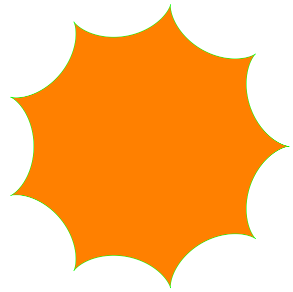
As responsible citicens, we **never ever** believe any AI blindly and confirm the numbers with other data. Wikipedia for example gives 8 times the volume, because it assumes that each solid has side length 2. The data match.

Problem 12) (10 points)

Find the **area** $\iint_R 1\, dA$ of the region R enclosed by the path

$$\vec{r}(t) = \begin{bmatrix} 8\sin(t) + \cos(8t) \\ 8\cos(t) + \sin(8t) \end{bmatrix}$$

where $0 \leq t \leq 2\pi$.



A useful tip: If you integrate an odd 2π periodic function from 0 to 2π you always get zero. Because the area below the graph is the area above.

Problem 13) (10 points)

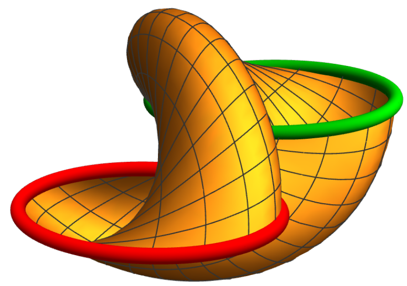
Find the flux $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ of the curl of the vector field

$$\vec{F}(x, y, z) = \begin{bmatrix} -y + zx^{10} \\ x + y \\ ze^{xy} \end{bmatrix}$$

through the surface S parametrized by

$$\vec{r}(t, s) = \begin{bmatrix} 2\cos(s)\sin(t) \\ s + 2\cos(t) \\ 2\sin(s)\sin(t)] \end{bmatrix},$$

where $0 \leq t \leq 2\pi$ and $0 \leq s \leq \pi$.



P.S. The surface is a twisted sphere. In order that your brain does not get twisted, we provide you with the following information: the boundary curve $\vec{r}(t) = \vec{r}(0, t)$ is oriented correctly while the boundary curve $\vec{r}(t) = \vec{r}(\pi, t)$ is oriented in an incompatible way to the surface S.