8/8/2024 FINAL EXAM **Maths 21a, O.Knill, Summer 2024**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are needed.

1) T F The graph $y = \sin(x)$ in the two dimensional plane is a curve parametrized by $\vec{r}(t) = [t, \sin(t)]$.

Solution:
Write $r(t) = [t, \sin(t)] = [x(t), y(t)]$ and check.

2) T F The vector fields $\text{curl}(\vec{F})$ and $\text{curl}(\vec{G})$ are perpendicular at every point, if $\vec{F}$ and $\vec{G}$ are perpendicular at every point in space.

Solution:
For example, $\vec{F} = z, z, 0, G = 0, 0, y$ are perpendicular but $\text{curl}(F) \cdot \text{curl}(G) = [-1, 1, 0] \cdot [1, 0, 0] = 1$ is not zero.

3) T F The curvature of the curve $\vec{r}(t) = [3\sin(t), 0, 3\cos(t)]$ is 3 everywhere.

Solution:
It is not a circle of curvature 1/3

4) T F Any function that simultaneously satisfies the Laplace and wave equation must be linear.

Solution:
Take $u(x, y) = xy$. This is not linear but a counter example. For "Laplace" is replaced by "heat" it is true. We might first think it is true because it implies $u_{xx} = 0$ and $u_{yy} = 0$ which suggests linearity. A "good student" concludes from this that u must be linear. But it is false as $u(x, y) = xy$ shows. I gave full credit for this problem in any case, because A) I got it first wrong also when writing the problem. And B), because "linear" could be interpreted as **multi-linear** which makes it true. A trained mathematician can sees xy as a bilinear form meaning that it is linear in each of the coordinates. It is not a linear function in the sense we have considered in the course, like when doing linearization. I also asked the question to AI's and they ALL got it wrong (and they well understood what linear means and they did not interpret it as multi-linear). They argued from $u_{xx} = 0, u_{yy} = 0$ that u must be linear, which is wrong as $u(x, y) = xy$ shows.

5) T F Maxwell took a calculus exam, where he had to prove Stokes theorem.

Solution:
Now it is correct. We had seen in an other exam a different order.

6) | T | | F | The boundary of disk surface $\{x^4 + y^4 \leq 1, z = 0\}$ is the curve $\{x^4 + y^4 = 1, z = 0$.

Solution:
Indeed. The boundary of a surface is a curve.

7) | T | | F | If $\vec{F}(x, y)$ is defined for all x, y and $\text{curl}(\vec{F}(x, y)) = 0$ everywhere, then $\vec{F}(x, y) = [f_x, f_y]$ is a gradient field.

Solution:
Indeed

8) | T | | F | In the movie "Rushmore", the "hardest problem of geometry" is solved using integration by parts.

Solution:
No, it was solved by substitution, trig substitution. Of course, to answer this question, you needed to be in class and have seen the clip. We discussed that it uses trig substitution.

9) | T | | F | We can use the fundamental theorem of line integrals to compute the area of a region.

Solution:
We use Green's theorem to compute the area or Stokes theorem to compute volume. The fundamental theorem of line integrals does not involve area.

10) | T | | F | The circles $[\cos(t), \sin(t), 0]$ and $[0, 1 + \cos(t), \sin(t)]$ are interlinked. You can not take them apart without them intersecting.

Solution:
If you draw the situation out, these two circles beautifully interlink. It is called a link.

11) T F The quartic $-x^2 + y^2 - (z-1)^2 = 4$ is a two-sheeted hyperboloid.

Solution:
It is indeed. Traces indicate that.

12) T F If $\vec{r}(\phi, \theta)$ is the standard parametrization of a sphere, then the tangent vectors $\vec{r}_\phi$ and $\vec{r}_\theta$ are perpendicular.

Solution:
Look at it geometrically. You see it also if you look at the standard grid curves. These lines of latitude and longitude are everywhere perpendicular on the sphere.

13) T F The gradient of the divergence of $\vec{F} = [P, Q, R]$ defines a vector field that is always perpendicular to $\vec{F}$.

Solution:
Not in general. It is indeed almost always false. Just take an example like $[0, 0, z^2]$ for which the gradient of the divergence is $[0, 0, 1]$ but that is not perpendicular to $[0, 0, z^2]$.

14) T F If $\vec{r}(t)$ parametrizes a curve, then the fourth derivative $\vec{r}\,''''(t)$ is called "snap".

Solution:
Indeed. We covered this in class.

15) T F For any bounded solid E in space, the boundary surface S has the property that it is a closed surface meaning that it does not have any boundary.

Solution:
This is a fundamental topological fact indeed.

16) T F Given two vectors $\vec{v}, \vec{w}$, define $\vec{u} = \vec{v} \times \vec{w}$. It is always true that $\vec{w}, \vec{u}$ are perpendicular.

Solution:
Since $\vec{u}$ is perpendicular to each of its factors.

17) T **F** If $\vec{v} = \text{curl}(\vec{F})/|\text{curl}(\vec{F})|$ is defined and $g(x, y, z) = \text{div}(\vec{F})$, then the directional derivative is $D_{\vec{v}}g = 0$.

Solution:
Let's unpack this. This claims that $\nabla g \cdot \vec{v} = 0$ which implies $\text{grad}(\text{div}(\vec{F}) \cdot \text{curl}(\vec{F}) = 0$. Take $F = [0, 0, xyz]$ for example, where this is $[1, 1, 0] \cdot [xz, -yz, 0] = xz - yz$. Virtually any random example would have worked to disprove this.

18) **T** F The vector projection of $\vec{F} = [P, Q, R]$ onto $[0, 0, 1]$ is $[0, 0, R]$.

Solution:
This is true for any vector and so also for a vector field.

19) **T** F For a given function f, define the new function $g = f_{xx}f_{yy} - f_{xy}^2$. Define $\vec{F} = \nabla g + \nabla f$, then the curl of $\vec{F}$ is zero.

Solution:
It is a gradient field!

20) **T** F The curvature of any curve on a sphere $x^2 + y^2 + z^2 = 1$ is always larger or equal to 1 at any point where the curvature is defined.

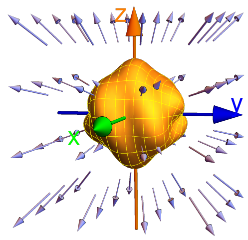
Solution:
One can see this intuitively: if you try to put an arc of a circle of radius larger than 1 onto the sphere, it will not fit. The best one can do to "straighten the curve" is to be on a grand circle, a curve of curvature 1. Any other curve will have larger curvature. Here is a more rigorous argument: we have seen that the curvature at a point is $1/r$ for a circle of radius r. For a general curve, it is the curvature of the best circle fitting locally near the point (called osculating circle). If we are on a sphere of radius 1, any circle on it has curvature larger or equal to one.

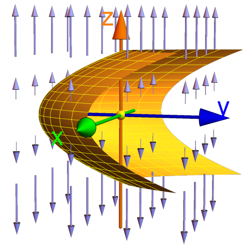
Problem 2) (10 points) No justifications are necessary.

a) (2 points) We decide in two cases whether the **flux** of the vector field through the surface S is [positive] or [negative]. In the left picture (belonging to the boxes to the left), S is oriented outwards, in the right picture S is oriented in the positive y direction.

The flux through the left surface is

Positive
Negative





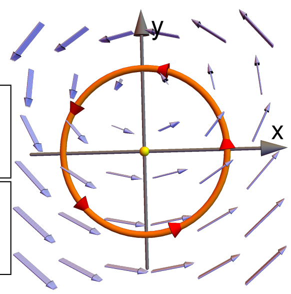
The flux through the right surface is

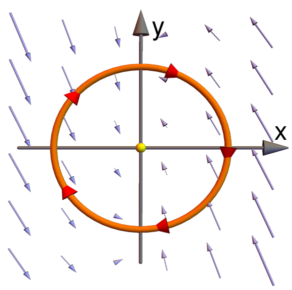
Positive
Negative

b) (2 points) Decide in each case, whether the **line integral** of the vector field along the closed circular loop is [positive] or [negative]. Check the boxes on each side.

The line integral along the curve is

Positive
Negative





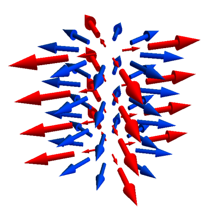
The line integral along the curve is

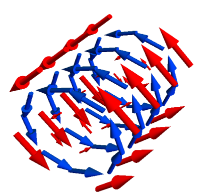
Positive
Negative

c) (2 points) Check on each side the box, for which the answer is positive [> 0]. Check exactly one box on the left for the left vector field and one box on the right for the right vector field.

For the field to the left: (check one box)

$\|\text{curl}(\vec{F})\| > 0$
$\text{div}(\vec{F}) > 0$





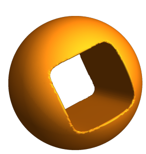
For the field to the right: (check one box)

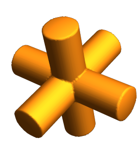
$\|\text{curl}(\vec{F})\| > 0$
$\text{div}(\vec{F}) > 0$

d) (2 points) Remember that a solid E is **simply connected** if every closed loop in E be pulled together continuously to a point within E. Which are simply connected?

The solid to the left is (check one box)

simply connected
not simply connected





The solid to the right is (check one box)

simply connected
not simply connected

e) (2 points) Write down the names of two partial differential equations involving functions $f(t,x)$ for which only the first derivative with respect to t appear.

Solution:
a) Positive, negative
b) Positive, negative
c) $\text{div}(\vec{F}) > 0$, $|\text{curl}(\vec{F})| > 0$.
d) not simply connected, simply connected.
e) There are several PDEs which could work: heat, Burger, transport, Black-Sholes, Schroedinger

Problem 3) (10 points)	No justifications are needed.

In a **2019 tictoc video** of Oliver, the **three fundamental derivative operations** div, curl and grad in multivariable calculus were investigated. There are 27 ways to combine three such operations. How many do make sense? In a) to c) you select the cases which **are defined**.

a) (3 points) In the following table, cross out every expression which is defined. Let $f(x, y, z)$ be a function of three variables.

grad(grad(grad(f)))	grad(curl(grad(f)))	grad(div(grad(f)))
curl(grad(grad(f)))	curl(curl(grad(f)))	curl(div(grad(f)))
div(grad(grad(f)))	div(curl(grad(f)))	div(div(grad(f)))

b) (3 points) In the following table, cross out every expression which **is defined**. Let $\vec{F} = [P, Q, R]$ denote a vector field in $\mathbf{R}^3$.

grad(grad(curl($\vec{F}$)))	grad(curl(curl($\vec{F}$)))	grad(div(curl($\vec{F}$)))
curl(grad(curl($\vec{F}$)))	curl(curl(curl($\vec{F}$)))	curl(div(curl($\vec{F}$)))
div(grad(curl($\vec{F}$)))	div(curl(curl($\vec{F}$)))	div(div(curl($\vec{F}$)))

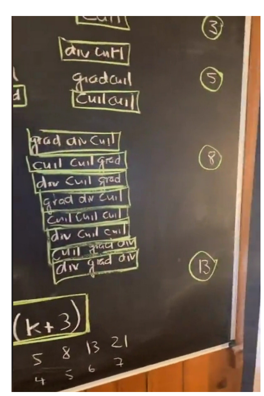
c) (3 points) In the following table, cross out every expression which **is defined**. Again, $\vec{F} = [P, Q, R]$ is a vector field in $\mathbf{R}^3$.

grad(grad(div($\vec{F}$)))	grad(curl(div($\vec{F}$)))	grad(div(div($\vec{F}$)))
curl(grad(div($\vec{F}$)))	curl(curl(div($\vec{F}$)))	curl(div(div($\vec{F}$)))
div(grad(div($\vec{F}$)))	div(curl(div($\vec{F}$)))	div(div(div($\vec{F}$)))

d) (1 point) Two of the following expressions are always zero (either the zero number or zero vector). Which ones? As before, $\vec{F} = [P, Q, R]$ is a vector field and $f(x, y, z)$ a scalar function in $\mathbf{R}^3$

curl(grad(f))	curl(curl($\vec{F}$))
div(grad(f))	div(curl(($\vec{F}$)

Oliver theorem from 2019 (see tictoc) tells that the number of ways in which one can combine n operations to make sense is given by the **Fibonacci number** $F(n + 3)$. In class, we had looked at the case $n = 2$, where $F(n + 3) = 5$ combinations made sense.

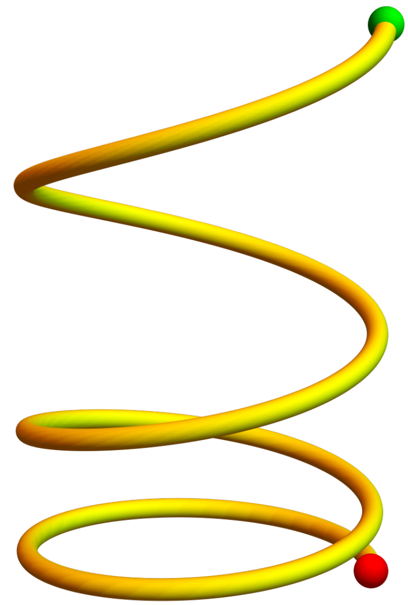


Screenshot from Oliver's Tictoc 2019

Solution:
Note that grad grad, and grad curl, and curl div and div div are combinations which do not work. This allows to discard many cases at the same time like the first column in a), the first column in b) and the second and third column in c).
a) curl curl grad, div curl grad, grad,div grad
b) curl curl curl, div curl curl grad div curl
c) curl grad div and div grad div
d) curl grad =0 and div curl=0.
By the way, there are $F(k+3)$ different ways to write a meaningful sentence in the *div*, *grad*, *curl* language, where $F(k)$ is the k'th Fibonacci number. For $k = 1$ there are 3, for $k = 2$ there are 5, for $k = 3$, there are 8. Oliver's tiktok video from December 2019 shows this using graph theory. It is very similar to the good will hunting story. We count paths in a graph of three nodes grad, div curl, where a connection from A to B is made if AB is possible. One has then the problem to count the number of paths of length k in this graph.

Problem 4) (10 points)

A curve $\vec{r}(t)$ satisfies $\vec{r}\,'(t) = \begin{bmatrix} 2\cos(3t) \\ 2\sin(3t) \\ t \end{bmatrix}$ has the initial point $\vec{r}(0) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.



a) (2 points) Locate the **end point** $\vec{r}(2\pi)$.

b) (2 points) Find the **acceleration** $\vec{r}\,''(0)$.

c) (2 points) Compute the **curvature** at $t = 0$.

d) (2 points) What is the **unit tangent vector** $\vec{T}(0)$?

e) (2 points) What is the **arc length** of the curve for $t \in [0, 2\pi]$?

Solution:

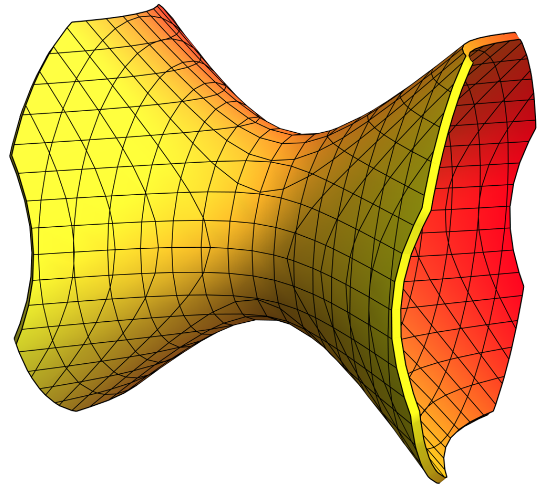
a) Integrate and adjust the constant. We have $\vec{r}(t) = [2\sin(3t)/3 + 3, -2\cos(3t)/3 + 4 + 2/3, t^2/2 + 5]$. And so $\vec{r}(2\pi) = [3, 4, 5 + 2\pi^2]$. b) $[0, 6, 1]$;

c) $|[2, 0, 0] \times [0, 6, 1]|/8 = \sqrt{37}/4$.

d) $[1, 0, 0]$.

e) $\int_0^{2\pi} \sqrt{1 + 4t^2}\, dt$. It can be integrated using integration by parts and gives $\pi\sqrt{1 + 16\pi^2} +$ $\text{arcsinh}(4\pi)/4$. We gave full credit for the integral. It is the same computation as for the arc length of the parabola. Indeed, unwrapping the curve from the cylinder to the plane, (as a student figured out) produces a parabola.

Problem 5) (10 points)



a) (2 points) Find the **tangent plane** to the hyperboloid $f(x, y, z) = x^2 + y^2 - z^2 = 1$ at $(1, 1, 1)$.

b) (2 points) Parametrize the **normal line** L to that plane passing through $(1, 1, 1)$.

c) (2 points) What is the **distance** of this normal line L to the origin $(0, 0, 0)$?

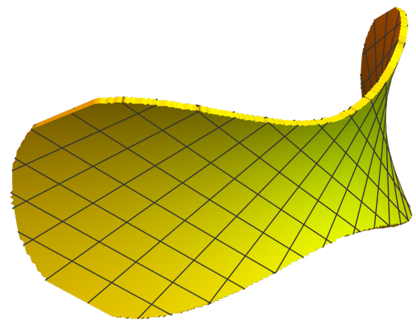
d) (2 points) **Estimate** the value $1.003^2 + 0.999^2 - 1.02^2$ using linear approximation (do not do it directly).

e) (2 points) What is the **directional derivative** $D_{\vec{v}} f$ at $(1, 1, 1)$ if $\vec{v} = \vec{i} = [1, 0, 0]$.

Solution:
a) The gradient is $[2x, 2y, -2z]$. At the point $(1, 1, 1)$ it is $[2, 2, -2]$. The equation of the plane is $2x + 2y - 2z = d$. Plugging in the point gives $2x + 2y - 2z = 2$.
b) $\vec{r}(t) = [1, 1, 1] + t[2, 2, -2]$.
c) It is $|[1, 1, 1] \times [2, 2, -2]|/|[2, 2, -2]| = 2\sqrt{2/3} = \sqrt{8}3$.
d) $1 + 2 \cdot 0.03 + 2 \cdot (-0.001) - 2 \cdot 0.02 = 0.964$.
e) $[2, 2, -2] \cdot [1, 0, 0] = 2$.

Problem 6) (10 points)

Find the surface area of the **travel sleep mask surface**



$$\vec{r}(u,v) = \begin{bmatrix} uv+1 \\ u+v \\ u-v+1 \end{bmatrix}$$

parametrized by the region $R = \{(u,v), u^2+v^2 \leq 2\}$.

Solution:
We have $|\vec{r}_u \times \vec{r}_v| = |[-2, x+y, -x+y]| = \sqrt{4+2x^2+2y^2}$. Integrating this over the region R is done using polar coordinates $\int_0^{2\pi} \int_0^1 \sqrt{4+2r^2} r \; dr d\theta = \frac{8\pi}{3}(2\sqrt{2}-1)$.

Problem 7) (10 points)

a) (5 points) Compute the double integral

$$\int_1^2 \int_0^{\sqrt{2-y}} \frac{\sin(\pi x)}{1-x^2} \, dxdy \, .$$

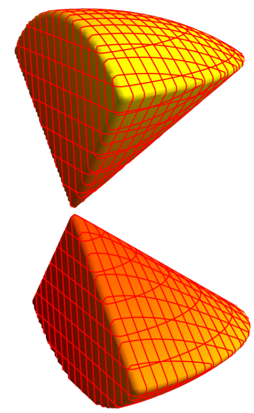
b) (5 points) Evaluate the integral

$$\iiint_E (x^2+y^2+z^2)^4 \, dV \, ,$$

where

$$E = \{(x,y,z), x^2+y^2+z^2 \leq 1, z^2 \geq x^2+y^2, y \geq 0\} \, .$$

The solid E is displayed to the right.



Solution:
a) Switch the order of integration. To do so, make a picture.

$$\int_0^1 \int_1^{2-x^2} \frac{\sin(\pi x)}{1-x^2} \, dydx \, .$$

Evaluating the inner integral gives $\frac{\sin(\pi x)}{1-x^2}[(2-x^2-1)] = \sin(\pi x)$. Integrating this from 0 to π gives $\frac{2}{\pi}$.
b) Use spherical coordinates: $2\int_0^\pi \int_{\pi/2}^\pi \rho^1 0 \sin(\phi) \, d\rho d\phi d\theta$ which gives $\frac{\pi}{11}(1-\frac{1}{\sqrt{2}})$.

Problem 8) (10 points)

Since we noticed some **log identity deficiencies** in the midterm, lets see whether we have made our homework and learned from our **mistakes**. Logs again!
Find the maxima and minima of

$$f(x, y) = \log(x) + 2\log(y) - x - y$$

on $x > 0, y > 0$ and classify them using the second derivative test. As usual $\log(x) = \ln(x)$ is the natural log.

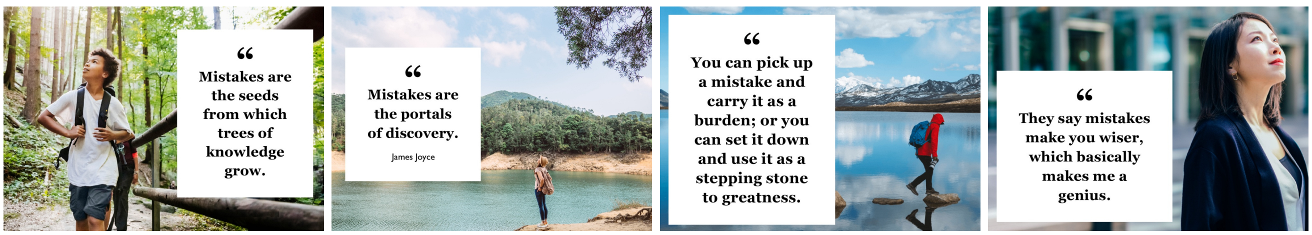


Image credit: **Kristin McKarty**, "35 quotes about learning from your mistakes to reassure you", Nov. 17, 2021, accessed from Lovetoknow.com, Aug. 6. 2024

Solution:
The gradient is $[1/x - 1, 2/y - 1]$ which is zero for $x = 1, y = 2$. There is only one critical point. We compute $D = 1/2, f_{xx} = -1$ so that this is a maximum. The maximal value is $2\log(2) - 3$

Problem 9) (10 points)

A Lagrange problem can be looked at from two different perspectives: either extremize f under the constraint $g = c$ or extremize g under the constraint $f = d$. It is a **Janus face**, a symbol of **duality**.

a) (5 points) Use the Lagrange method to find the maximum of

$$f(x, y) = xy$$

under the constraint

$$g(x, y) = x + 3y = 6 .$$

b) (5 points) Now use the Lagrange method to find the minimum of

$$f(x, y) = x + 3y$$

under the constraint $xy = 3$.

Statue of Janus, the god of all beginnings, gates, transitions, time, choices, duality, doorways, passages and endings. Image source: Wikipedia

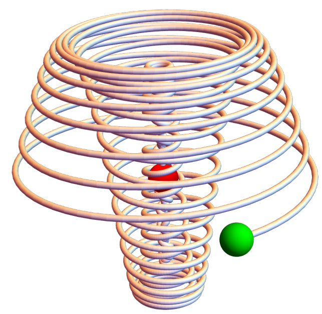
Solution:
a) This is a routine Lagrange problem. The Lagrange equations $y = \lambda x, x = \lambda 3, x+3y = 6$ has the only solution $(3, 1)$.
b) In this dual problem there are two solutions $(-3, -1)$ and $(3, 1)$. The minimum is $(-3, -1)$ the maximum is $(3, 1)$.

Problem 10) (10 points)

This year features the movie **"twisters"**, which has been quite well received and which Oliver still wants to see, once summer school is over. Compute the line integral

$$\int_C \vec{F} \cdot \vec{dr}$$

along the **twister curve** C parametrized as



$$\vec{r}(t) = \begin{bmatrix} e^{-t}\cos(100t) \\ e^{-t}\sin(100t) \\ e^{-t/100}\sin(t) \end{bmatrix}$$

from $t = 0$ to $t = \infty$. It connects $A = r(\vec{0})$ with $B = \vec{r}(\infty) = (0, 0, 0)$. The vector field is

$$\vec{F}(x, y, z) = \begin{bmatrix} 10x^4 \\ 6y^2 + z \\ 3z^2 + y \end{bmatrix} .$$

Solution:
Use the fundamental theorem of line integral. The potential is $f(x, y, z) = 2x^5 + 2y^3 + zy + z^3$. Now $f(B) - f(A) = f(0, 0, 0) - f(1, 0, 0) = 2$.

Problem 11) (10 points)

We place the **Stanford bunny** E into a radiation field $\vec{F}$. The bunny is the most famous object in computer science. Its skin S is a closed surface. We assume here that S is oriented **inwards**. You enter the Mathematica command line E=DiscretizeGraphics[ExampleData[{"Geometry3D", "StanfordBunny"}]; NIntegrate[1,{x,y,z} ∈ E] which gives you the output 0.057. What is the flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

of the field

$$\vec{F}(x,y,z) = \begin{bmatrix} 500x + y + z \\ 100x + 100z \\ x + y + 500z \end{bmatrix}$$

through the **Stanford Bunny surface** S?

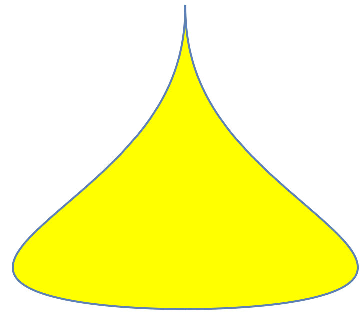
Solution:
This is a problem for the divergence theorem. The divergence is constant 1000. The orientation however is opposite and $-\int_S \vec{F}\ d\vec{S} = -\iiint_E \text{div}(\vec{F})\ dV = -\iiint_E 1000\ dV = -1000\text{Vol}(E) = 1000 \cdot 0.057 = \boxed{-57}$.
P.S.The funny thing is that Mathematica actually computes the volume using Stokes theorem. The bunny can be downloaded (just google "stanford bunny) as an STL file. This essentially contains triples of points $A = (A_1, A_2, A_3)$,$B = (B_1, B_2, B_3)$, $C = (C_1, C_2, C_3)$, as well as the normal vector $n = \vec{AB} \times \vec{AC} = [a, b, c]$. The area of the triangle is $|\vec{AB} \times \vec{AC}|/2$. The flux of $\vec{F} = [0, 0, z]$ through the triangle is given as follows: parametrize the surface as $\vec{r}(u,v) = A + t\vec{AB} + s\vec{AC}$ and compute $\int_0^1 \int_0^1 \vec{F}(\vec{r}(u,v)) \cdot \vec{n}\ dudv/2 = \int_0^1 \int_0^1 A_3 + u(B_3 - A_3) + v(C_3 - A_3)c\ du/2 = A_3c/2 + (B_3 - A_3)c/4 + (C_3 - A_3)c/4$ which is $c(B_3 + C_3)/4$. We see that we can add all these values for all triangles and get the volume of the bunny.

Problem 12) (10 points)

Find the area of the **honey-drop region** enclosed by

$$\vec{r}(t) = \begin{bmatrix} \frac{\sin^2(\pi t)}{t} \\ \\ 4t^2 - 4 \end{bmatrix}$$

parametrized on the interval $-1 \leq t \leq 1$.



Solution:
Use Green's theorem with $\vec{F} = [0, x]$, a vector field with constant curl 1. The anwer is $\int_{-1}^{1} \vec{F}(r(t)) \cdot \vec{r}\,'(t)\ dt$. This is $\int_{-1}^{1} [0, \frac{\sin^2(\pi t)}{t}] \cdot [\frac{d}{dt} \frac{\sin^2(\pi t)}{t}, 8t] = \int_{-1}^{1} 8 \sin^2(\pi t)\ dt = 8 \int_{-1}^{1} (1 - \cos(2\pi t)/2\ dt = 8$.

Solution:

Problem 13) (10 points)

Find the flux

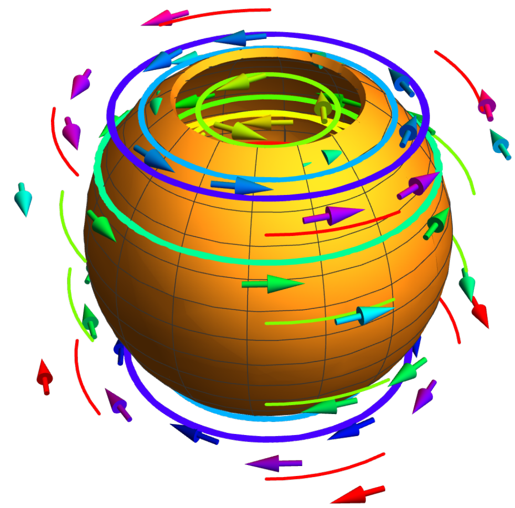
$$\iint_S \text{curl}(\vec{F}) \cdot \vec{dS}$$

of the curl of the vector field

$$\vec{F}(x,y,z) = \begin{bmatrix} -yz^2 \\ xz^2 \\ x^4y^6z^9\sin(e^{xyz}) \end{bmatrix}$$

through the outwards oriented surface

$$S = \{(x,y,z), |\ x^2+y^2+z^2 = 4 \text{ and } z \leq \sqrt{3}\}\ .$$



Solution:
This is a Stokes theorem problem. Instead of the flux $\iint_S \text{curl}(\vec{F})\ dS$ we compute the integral along the boundary. The boundary of the surface is a circle $r = 4 - \sqrt{3}^2 = 1$ parametrized by $\vec{r}(t) = [\cos(t), -\sin(t), \sqrt{3}]$. Note that we had to change the orientation in order that the surface is to our left. The line integral is

$$\int_0^{2\pi} [\sin(t)3, \cos(t)3, \cos^4(t)\sin^6(t)\sqrt{3}^9\sin(e^{-\cos(t)\sin(t)\sqrt{3}})] \cdot [-\sin(t), -\cos(t), 0]\ dt = -6\pi$$