7/20/2023 SECOND HOURLY Practice 2 Maths 21a, O.Knill, Summer 2023

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F

The gradient of the function $f(x, y) = x^2 + y^2$ at a point $(1, 1)$ is perpendicular to the gradient of $g(x, y) = x^2 - y^2$ at $(1, 1)$.

Solution:

Just compute the gradients at the point. It is $[2, 2]$ and $[2, -2]$

2) T F

The function $f(x, y) = xy$ has only one critical point $(0, 0)$.

Solution:

The gradient is $[y, x]$ which is zero only at $(x, y) = (0, 0)$.

3) T F

If the discriminant D and f_{xx} are positive at a critical point $(1, 1)$, then $(1, 1)$ is a minimum.

Solution:

This is part of the second derivative test.

4) T F

The integral $\int_0^\pi \int_0^5 1 \, dr d\theta$ is the area of a half disk of radius 5.

Solution:

Just to test your awareness of the r factor.

5) T F

If $f_{xy} = f_{xx} = f_{yy}$ at a critical point $(0, 0)$ then the discriminant D must be negative at $(0, 0)$.

Solution:

The discriminant is then $f_{xx} f_{yy} - f_{xx}^2 = 0$.

6) T F

The function $L(x, y) = (2x)(x - 1) + (4y)(y - 2)$ is the linearization of $f(x, y) = x^2 + 2y^2$ at the point $(1, 2)$.

Solution:

The linerization is never a non-linear function.

7) T **F** The gradient of $f(x,y) = x^3+y^3$ at the point $(1,1)$ is a vector perpendicular to the surface $z = x^3 + y^3$ at the point $(1,1,2)$.

Solution:

Nonsense. We have completely different objects. The gradient of f is a vector in two dimensions. The vector perpendicular to the surface $z = x^3 + y^3$ is a vector in three dimension. It is a classic confusion we have addressed once also in class.

8) T **F** Assume f and g are functions which both have the same critical point $(0,0)$ then f must be a multiple of g.

Solution:

Take $f = x^2 + y^2$ and $g = xy$. They are not multiples of each other.

9) **T** F Assume $\vec{r}(t)$ is a planar curve with constant speed $|\vec{r}\,'(t)| = 1$ and $f(x,y)$ a function, then $d/dt f(\vec{r}(t)) = D_{\vec{r}\,'(t)} f(\vec{r}(t))$.

Solution:

Yes, this is the chain rule.

10) **T** F If $f_{yy}(x,y) > 0$ for all points (x,y), then f can not have any local maximum.

Solution:

We would have $f_{yy} < 0$ at a local minimum.

11) **T** F If $f_{xx} = 0$ for all x, y, then at every critical point of f with $D \neq 0$, we have a saddle point.

Solution:

The discriminant is then $-f_{xy}^2$ which is non-positive. As it is non-zero it has to be negative.

12) T **F** If $(0,0)$ is saddle point for a function f, then there are directions $\vec{v}, \vec{w}$ such that $D_{\vec{v}} f(0,0) > 0, D_{\vec{w}} f(0,0) < 0$.

Solution:

A classic also. The directional derivatives are zero at a critical point.

13) **T** F $f_{tt} = f f_{xxx}$ is an example of a partial differential equation.

Solution:

This is a probably unstudied differential equation. Lets call it the Knill equation. (You can come up with your own PDE and name it. Seriously, usually you have to earn your name on a PDE. If I would spend half of my life studying this equation and find some interesting behavior and results about it, maybe it would be a path to eternity ...

14) T **F** The Fubini identity assures that $\int_0^1 \int_1^3 f(x,y)\, dydx = \int_1^3 \int_0^1 f(x,y)\, dydx$.

Solution:

One has also to switch $dxdy$.

15) **T** F Assume $(0,0)$ is not a critical point of f. Then the direction of steepest increase of f at $(0,0)$ is $\nabla f(0,0)/|\nabla f(0,0)|$.

Solution:

This is a basic property

16) T **F** If $(0,0)$ is a maximum of a function $f(x,y)$ under the constraint $g(x,y) = 0$, then either $(0,0)$ is a critical point of g or then a critical point of f.

Solution:

It is in general neither. Even the problem of this exam could be rewritten as such by putting the constant to the other side.

17) | T | F |

T (highlighted)

The linearization of $f(x,y) = 3x + 4y + 5$ at the point $(1,1)$ is $L(x,y) = 3x + 4y + 5$.

Solution:

The linearization of a linear function is the function itself.

18) | T | F |

F (highlighted)

If $f(x,y)$ has a maximum at $(0,0)$, then $D > 0$ and $f_{xx} < 0$.

Solution:

Not necessarily. An counter example is $f(x,y) = -x^4 - y^4$.

19) | T | F |

T (highlighted)

The area the region $4 \leq x^2 + y^2 \leq 9, x \leq 0$ is $\int_{\pi/2}^{3\pi/2} \int_2^3 r \, dr d\theta$.

Solution:

All set up correctly. Also with the integration factor r.

20) | T | F |

F (highlighted)

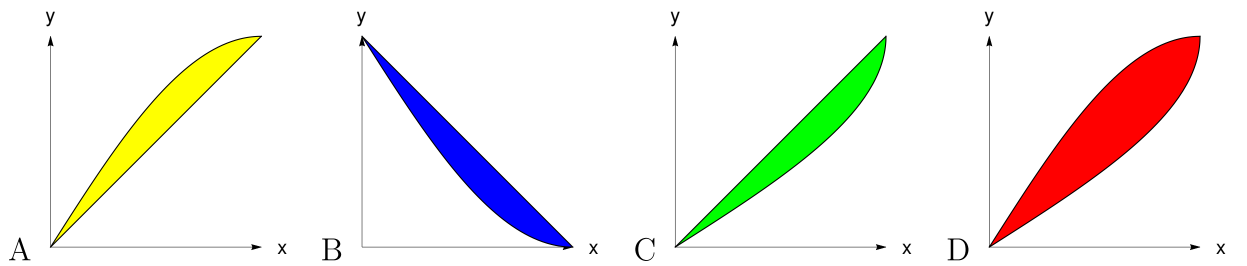
For any invertible function $0 \leq g(x) \leq 1$ we have $\int_0^1 \int_{g(x)}^1 f(x,y) \, dydx = \int_0^1 \int_0^{g^{-1}(y)} f(x,y) dxdy$.

Solution:

Simple examples like $g(x) = x/2$ provide counter examples. In order to make it true, we would need more conditions. An example which would work is if $g(0) = 0, g(1) = 1$ and $g(x)$ is monotone.

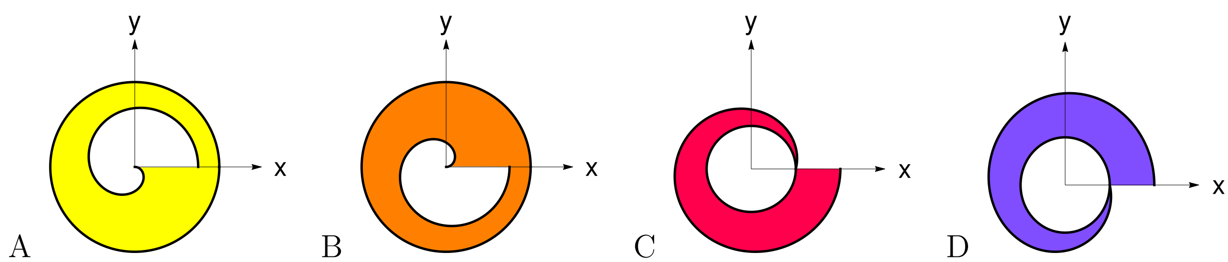
Problem 2) (10 points) No justifications are needed in this problem.

a) (4 points) Match the regions with their area formulas. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^1 \int_{2\arcsin(x)/\pi}^{x} 1 \; dydx$
	$\int_0^1 \int_{1-\sin(\pi x/2)}^{1-x} 1 \; dydx$
	$\int_0^1 \int_{2\arcsin(x)/\pi}^{\sin(\pi x/2)} 1 \; dydx$
	$\int_0^1 \int_{x}^{\sin(\pi x/2)} 1 \; dydx$

b) (4 points) Match the regions with their area integrals. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^{2\pi} \int_2^{2+2\sin(t/4)} r \; drdt$
	$\int_0^{2\pi} \int_2^{2+2\cos(t/4)} r \; drdt$
	$\int_0^{2\pi} \int_{3\sin(t/4)}^{4} r \; drdt$
	$\int_0^{2\pi} \int_{3\cos(t/4)}^{4} r \; drdt$

c) (2 points) We want you to write down the formulas for two partial differential equations for the unknown function $f(x,t)$.

Wave equation: [ ]

Black-Scholes equation: [ ]

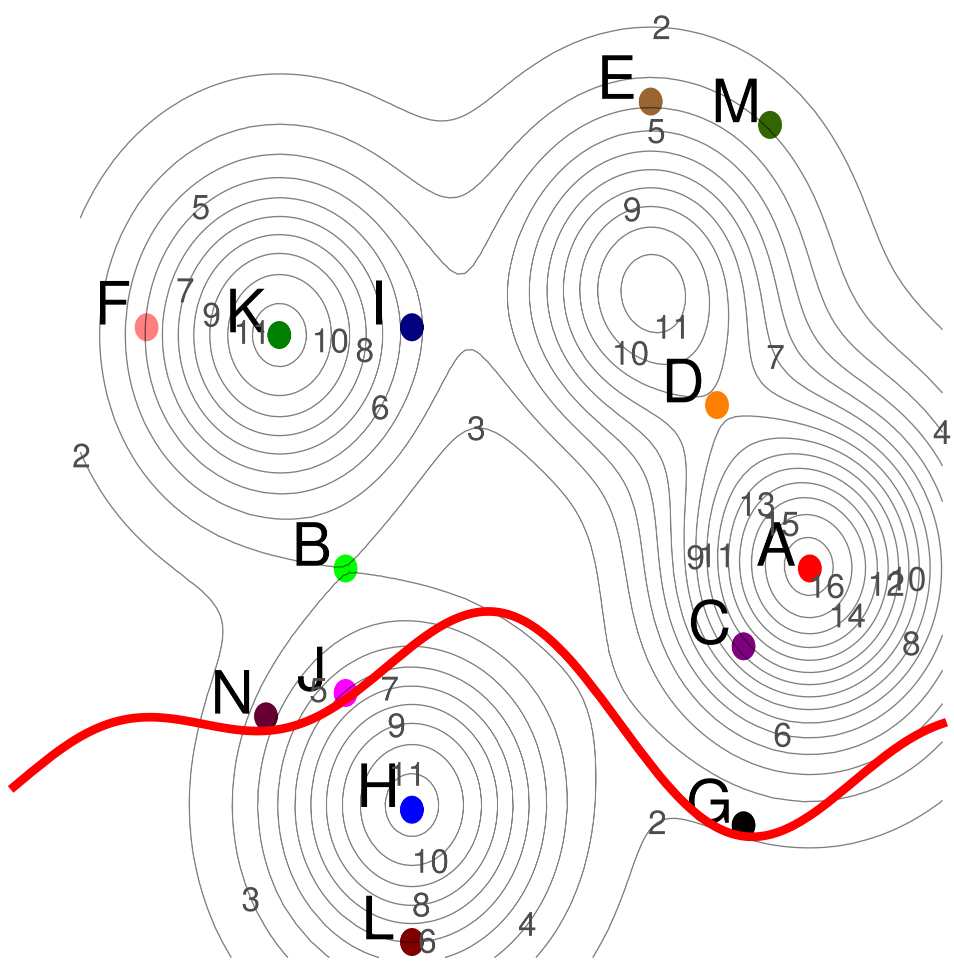
Solution:

a) CBDA
b) CDBA
c) $u_tt = u_xx$, $f_t = f - xf_x - x^2 f_{xx}$.

Problem 3) (10 points) No justifications are needed in this problem.

(10 points) We see the contours of an unknown smooth function, $f(x, y)$. The thick red curve is a constraint $g(x, y) = 0$. Use each of the labels A-N only once. Read carefully: in the last two questions, we want you to enter two letters You will therefore use 12 of the 14 letters $A - N$ and none of them twice.

	Enter A-N here
The point among A-N with maximal $\|\nabla f\|$.	
A point, where $f_x < 0, f_y = 0$.	
A point, where $f_x > 0, f_y = 0$.	
A point, where $f_y > 0, f_x = 0$.	
A point, where $f_y < 0, f_x = 0$.	
A global maximum of $f(x, y)$.	
A local maximum of $f(x, y)$ on $\{g(x, y) = 0\}$.	
A local minimum of $f(x, y)$ on $\{g(x, y) = 0\}$.	
Two points that are saddle points	and
Two points that are local but not global max	and



Solution:

CIFLEAJG (B AND D) (K and H).

Problem 4) (10 points)

a) (8 points) Classify the critical points of the function

$$f(x,y) = x^{10} - 5x^2 + y^2 + 2y$$

using the second derivative test.

Point	D	f_{xx}	nature

b) (2 points) Is there a global maximum or minimum of $f(x,y)$? (No explanation is necessary for this part b).)

	Yes	No
There is a global max for f		
There is a global min for f		

Solution:

a)

x	y	D	f_xx	Type	f
0	-1	-20	-	saddle	-1
-1	-1	160	80	minimum	-5
1	-1	160	80	minimum	-5

b) There is a global minimum (the x^{10} and y^2 terms dominate, but no global maximum. For x=0, one has $y^2 + 2y$ which is growing indefinitely for $y \to \infty$.

Problem 5) (10 points)

Use the Lagrange method to solve the problem to extremize

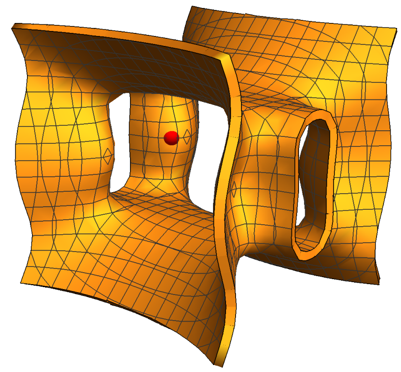
$$f(x,y) = 5 + x^3 + y^3$$

under the constraint $g(x,y) = 9x + 4y = 35$. There is only one solution with positive x and with positive y. Find this solution.

Solution:

The Lagrange equations are the three equations $3x^2 = \lambda 9, 3y^2 = \lambda 4, 9x + 4y = 35$ for the 3 unknowns x, y, λ. Eliminating λ gives $4x^2 = 9y^2$. Since we were told that x, y are positive, we can plug in $2x = 3y$ into the third equation and get $x = 3, y = 2$. This is the solution. (In general, we also would have to consider $2x = -3y$ but would get a bit more complicated fractions.)

Problem 6) (10 points)



a) 5 points) Find the equation $ax + by + cz = d$ of the tangent plane to the surface

$$f(x, y, z) = x^4 - y^4 + z^2 + x^2y^2 - x^2z^2 + y^2 - z^2 = -3$$

at the point $(1, 0, 2)$.

b) (5 points) Estimate $f(1.01, 0.02, 1.97)$ using linearization.

Solution:

a) The gradient is $\nabla f = [-4, 0, -4] = [a, b, c]$. The equation therefor is $-4x - 4z = d$. Plugging in the point $(1, 0, 2)$ gives the tangent plane $-4x - 4y = -12$ which i$x + z = 3$.
b) The estimation is $f(1, 0, 2) + a(x - 1.01) + b(y - 0.02) + c(z - 1.97)$ which is $-3 - 0.04 + 0.12 = -2.92$.

Problem 7) (10 points)

Find the **surface area** of the surface

$$\vec{r}(u, v) = [2u, 4 - u^2 - v^2, 2v]$$

with parameters satisfying $u^2 + v^2 \leq 25$.

Solution:

We have $\vec{r}_u \times \vec{r}_v = [-4u, -4, -4v]$ has length $4\sqrt{1 + u^2 + v^2}$. The integral $\int \int_R 4\sqrt{1 + u^2 + v^2} dudv$ is best done in polar coordinates $\int_0^{2\pi} \int_0^5 4\sqrt{1 + r^2} r \quad drd\theta$. The result is $\frac{8\pi}{3}(26^{3/2} - 1)$.

Problem 8) (10 points)

a) (5 points) You know $D_{\vec{v}}f(1,2) = \sqrt{2}$ for $\vec{v} = [1,1]/\sqrt{2}$. You also know $D_{\vec{w}}f(1,2) = 1$ for $\vec{w} = [3,4]/5$. Find the directional derivative $D_{\vec{u}}f(1,2)$ for $\vec{u} = [1,-1]/\sqrt{2}$.
b) (5 points) Write down the equation $ax + by = d$ of the **tangent line** to the level curve $f(x,y) = f(1,2)$ at the point $(1,2)$, where $f(x,y)$ refers to the same function than in a).

Solution:

a) Write $\nabla f = [a,b]$. We know $a + b = 2$ and $3a + 4b = 5$. This gives $[a,b] = [3,-1]$. Now t$[3,-1].[1,-1]/\sqrt{2} = 4/\sqrt{2} = 2\sqrt{2}$.
b) The equation is $3x - y = d$, where the constant d is obtained by plugging in at the point $1,2$ which is $3x - y = 1$.

Problem 9) (10 points)

a) (5 points) Evaluate the following double integral

$$\iint_G \frac{1}{(x^2+y^2)^3}\, dxdy \ ,$$

where G is region given by

$$\{4 \leq x^2 + y^2 \leq 9, y < 0\} \ .$$

b) (5 points) As usual, $\log = \ln$ is the natural log. Compute the following integral:

$$\int_1^e \int_0^{\sqrt{\log(x)}} \frac{12x}{e^2 - e^{(2y^2)}}\, dydx \ .$$

Solution:

a) Of course polar coordinates $\pi \int_2^3 (1/r^6) r\, dr = \pi(1/2^5 - 1/3^5)$.
b) Switch the order of integration using a picture of the region. Note that $y = \sqrt{\log(x)}$ means $x = e^{y^2}$. The left to right integral is now $\int_0^1 \int_{e^{y^2}}^e 12x/(e^2 - e^{2y^2}) dx\, dy$. This cancels nicely and the result is 6.