7/20/2023 SECOND HOURLY **Maths 21a, O.Knill, Summer 2023**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

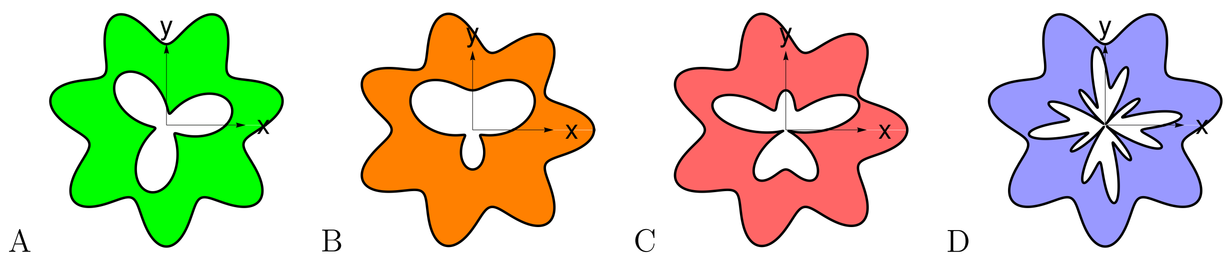
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1)	T	F	If $\nabla f(0,0)$ is non-zero then $\vec{v} = \nabla f(0,0)/|\nabla f(0,0)|$ is a unit vector for which $D_{\vec{v}}(f) = |\nabla f|^2$ is positive.
2)	T	F	The integral $\iint_R f(x,y)\, dA$ is the volume under the graph of a function and so always bigger or equal to zero.
3)	T	F	We can use multi-variable calculus to compute the improper integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx$.
4)	T	F	It is possible that a function $f(x,y)$ has a local maximum where $f_{xx} = -f_{yy}$.
5)	T	F	$(0,0)$ is a local maximum of the function $f(x,y) = 5 - x^{88} - y^{88}$.
6)	T	F	Assume $(1,1)$ is a solution of the Lagrange equations for $f(x,y)$ under the constraint $g(x,y) = 0$. If λ is negative then $(1,1)$ is a saddle point.
7)	T	F	The chain rule assures that $\frac{d}{dt}\vec{r}(f(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(f(t))$ if f is a function of three variables and $\vec{r}(t)$ parametrizes a curve.
8)	T	F	If R is the unit disk $x^2 + y^2 \leq 1$, then $\iint_R x^2 + y^2\, dA \leq \iint_R 1\, dA$.
9)	T	F	If $f(x,y,z) = x^2yz + y^3xz^2 + y^5 = 4$ defines z as a function $z = g(x,y)$ near the point $(1,1,1)$ then by implicit differentiation, $g_x(1,1) = -f_x(1,1,1)/f_z(1,1,1)$.
10)	T	F	If $(1,1)$ is a critical point of f and $f_{xx}(1,1) = 1$ and $f_{yy}(1,1) = 1$, then $(1,1)$ is a local minimum.
11)	T	F	If the discriminant at a critical point is non-zero, then we know the critical point is either a local maximum, a local minimum or a saddle point.
12)	T	F	If $R = \{x > 0\}$ then $\iint_R x\, dxdy = \int_0^{\pi} \int_0^{\infty} r\cos(\theta) r drd\theta$.
13)	T	F	By linear approximation, we can estimate $\sqrt{102.4} = 10 + 2.4/20 = 10.12$.
14)	T	F	If $(3,3)$ is a critical point of $f(x,y)$, then $(3,3)$ is also a critical point for the function $f(x,y)^2$.
15)	T	F	The gradient of $f(x,y)$ is a vector perpendicular to the graph $z = f(x,y)$.
16)	T	F	If (x_0, y_0) is a min of $f(x,y)$ then (x_0, y_0) is a minimum under the constraint $g(x,y) = c$.
17)	T	F	The area of $x^4 + y^4 \leq 4$ is larger or equal than the area of the region $x^2 + y^2 \leq 4$.
18)	T	F	If $\vec{v}$ is a unit vector perpendicular to $\nabla f(1,1,1)$ and $(1,1,1)$ is not critical point of f, then $D_{\vec{v}} f(x,y,z) = 0$.
19)	T	F	The vector $\vec{r}_u(1,1)$ is tangent to the surface parametrized by $\vec{r}(u,v)$.
20)	T	F	There is a function $f(x,y)$ which is continuous on the open square $(0,1) \times (0,1)$ but not continuous on $[0,1] \times [0,1]$ where Fubini fails.

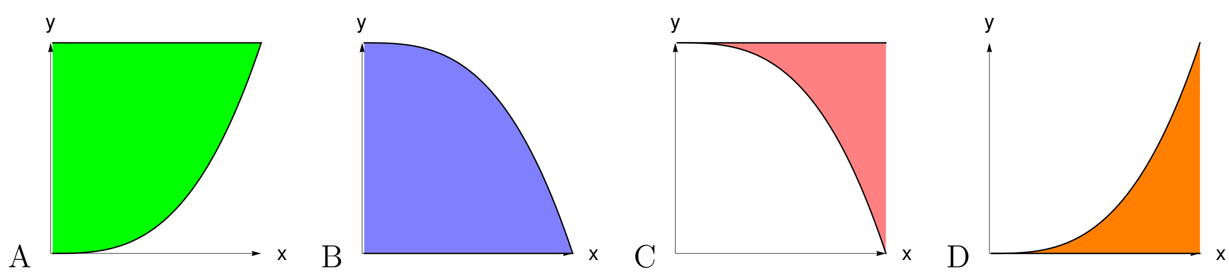
Problem 2) (10 points) No justifications are needed in this problem.

a) (4 points) Please match **polar regions** with area integrals. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^{2\pi} \int_{2+\sin(5t)+\sin(3t)}^{5+\cos(7t)} r\,dr d\theta$
	$\int_0^{2\pi} \int_{2+\sin(12t)+\cos(4t)}^{5+\sin(7t)} r\,dr d\theta$
	$\int_0^{2\pi} \int_{2+\sin(3t)+\sin(t)}^{5+\cos(7t)} r\,dr d\theta$
	$\int_0^{2\pi} \int_{2+\sin(3t)+\cos(3t)}^{5+\sin(7t)} r\,dr d\theta$

b) (4 points) Now match the regions with the area formulas. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^1 \int_{y^{1/3}}^1 1\,dx dy$
	$\int_0^1 \int_0^{1-x^3} 1\,dy dx$
	$\int_0^1 \int_0^{y^{1/3}} 1\,dx dy$
	$\int_0^1 \int_{1-x^3}^1 1\,dy dx$

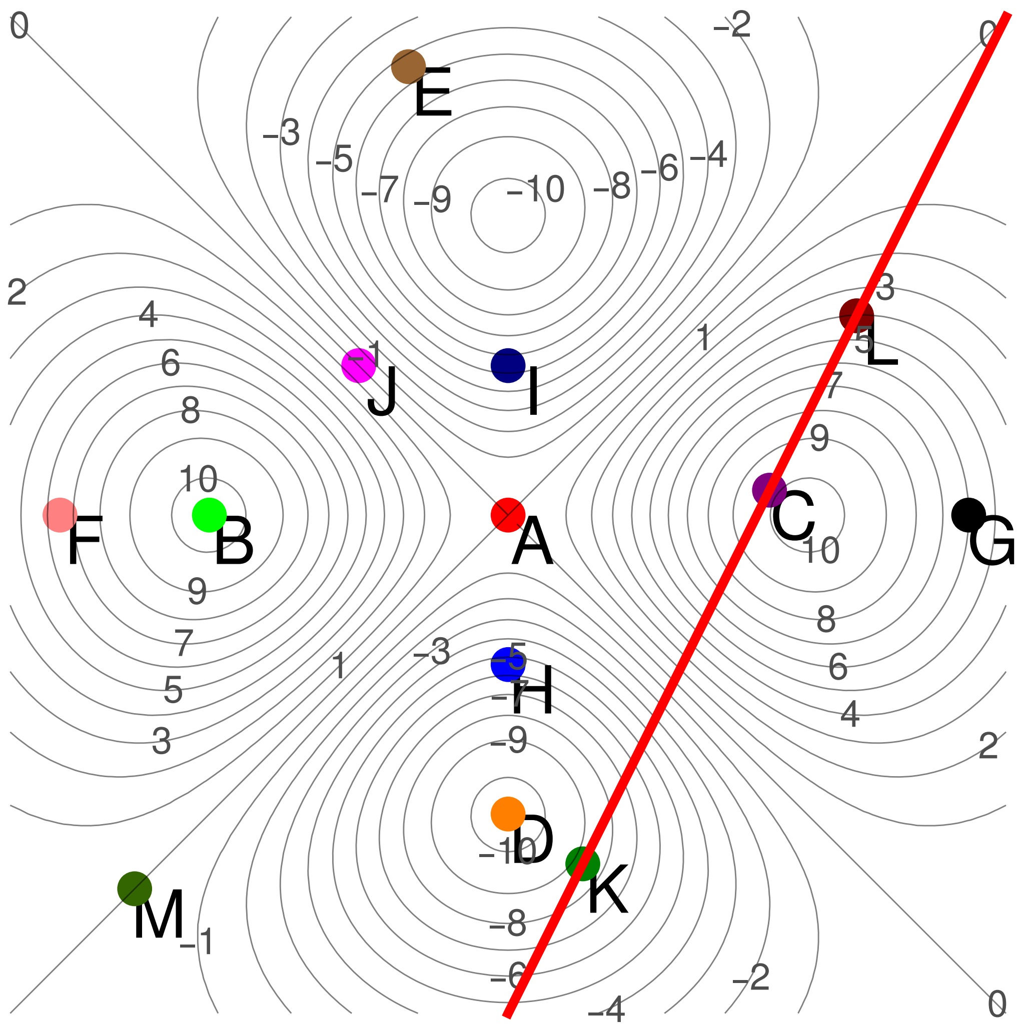
c) (2 points) Recall the differential equations for the unknown function of two variables.

Wave equation for $f(t,x)$:

Laplace equation for $f(x,y)$:

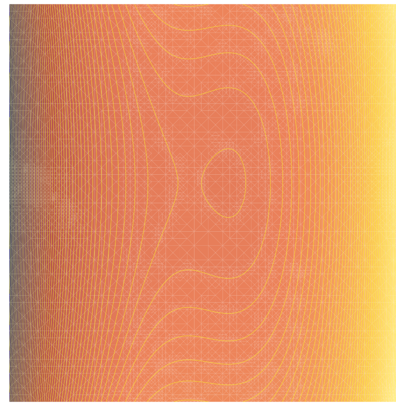
Problem 3) (10 points) No justifications are needed in this problem.

Answer the following 10 questions for the **contour map** of a function $f(x, y)$. There is a line drawn which passes through the points K, C, L. This line is referred to as $g(x, y) = 0$ in the last two problems. Each problem is worth one point. Pick 10 of the 12 locations A-M on the map.



	Enter A-M
A point, where $f_x < 0, f_y = 0$.	
A point, where $f_x > 0, f_y = 0$.	
A point, where $f_y > 0, f_x = 0$.	
A point, where $f_y < 0, f_x = 0$.	
A saddle point of $f(x, y)$.	
A local minimum of $f(x, y)$.	
A local maximum of $f(x, y)$.	
A non critical point among A-M with minimal $\lvert\nabla f\rvert > 0$.	
A local maximum of $f(x, y)$ on $\{g(x, y) = 0\}$.	
A local minimum of $f(x, y)$ on $\{g(x, y) = 0\}$.	

Problem 4) (10 points)



Elliptic curves are important in many fields like cryptography, analysis or physics.

a) (8 points) Classify the critical points of the function

$$f(x, y) = x^3 + y^2 - 3x - 2y$$

using the **second derivative test**.

b) (2 points) Is there a global max or min for f? If yes, locate them. Otherwise, tell why they do not exist.

Level curves of the function f are called elliptic curves.

Problem 5) (10 points)

You have seen the analog of the problem for a dice where we had 6 variables. We do it here for a coin, where x is the probability that the coin shows head and y is the probability that the coin shows tail. Use the Lagrange method to solve this problem. Your task is to find the probability distribution (x, y) that maximizes the **Shannon entropy**

$$f(x, y) = -x \log(x) - y \log(y)$$

under the constraint $g(x, y) = x + y = 1$. As usual, we wrote $\log(x) = \ln(x)$ for the natural log.

Problem 6) (10 points)

a) (3 points) Find the **tangent line** to the curve

$$f(x, y) = x^2 + xy + y^2 = 3$$

at the point $(1, 1)$.
b) (4 points) Find the **tangent plane** to the **air tag level surface**

$$f(x, y, z) = x^2 + xy + y^2 + yz + z^2 = 5 ,$$

at the point $(1, 1, 1)$.
c) (3 points) Estimate $f(1.01, 1.001, 1.0001)$ for the air tag function using linear approximation.

Oliver's wallet and keys are air tagged with ellipsoids

Problem 7) (10 points)

Use the standard surface area formula to compute the surface area of the surface which has the parametrization

$$\vec{r}(u, v) = \begin{bmatrix} u^2 + v^2 \\ u^2 - v^2 \\ u^2 \end{bmatrix}$$

and where the parameter domain is $0 \leq u \leq 1$ and $0 \leq v \leq 1$.

Problem 8) (10 points)

In order to honor the magic of the number 21, we evaluate the double integral

$$\int_1^2 \int_0^{4-x^2} \frac{y^{21}}{\sqrt{4-y} - 1} \, dy \, dx .$$

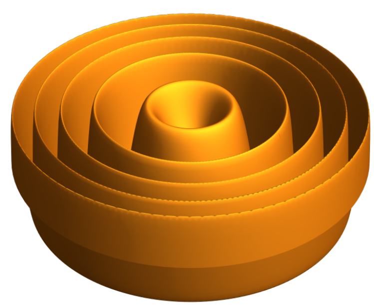
Problem 9) (10 points)

Evaluate the double integral

$$\iint_G 2\pi \sin(\pi(x^2 + y^2)) \, dxdy ,$$

where G is region given by

$$\{0 \leq x^2 + y^2 \leq 9, y > |x|\} .$$



The graph of $2\pi \sin(\pi(x^2 + y^2))$.