7/6/2023 FIRST HOURLY **Maths 21a, O.Knill, Summer 2023**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F The curvature of $\vec{r}(t)$ is a vector perpendicular to the curve.

Solution:

The curvature is a scalar

2) T F $\vec{v} \times (\vec{w} \times \vec{v}) = \vec{0}$ for all vectors $\vec{v}$ and $\vec{w}$.

Solution:

Take $v = \vec{i}$ and $w = \vec{j}$.

3) T F The vector projection $\vec{P}$ is always commutative in the sense $\vec{P}_{\vec{v}}(\vec{P}_{\vec{w}}(\vec{u})) = \vec{P}_{\vec{w}}(\vec{P}_{\vec{v}}(\vec{u}))$.

Solution:

The first vector is parallel to $\vec{v}$, the second parallel to $\vec{w}$.

4) T F If a vector $\vec{v}$ has integer components, then $\vec{v} \cdot \vec{v}$ is an integer.

Solution:

It is a sum of products of integers.

5) T F The function $f(x, y) = xy/(x^4 + y^4)$ is continuous at $(0, 0)$.

Solution:

Taking polar coordinates shows that it is not.

6) T F If $\vec{u}, \vec{v}$ form an acute angle, then $\vec{u}, -\vec{v}$ form an obtuse angle.

Solution:

Draw it out. We have complementary angles.

7) T **F** We have $|\vec{v} \cdot \vec{w}| \leq |\vec{v}|^2$ for all vectors $\vec{v}, \vec{w}$.

Solution:

Cauchy Schwartz would give $|\vec{v}||\vec{w}|$ on the right hand side.

8) **T** F The curvature of the curve $\vec{r}(t) = [2\cos(t), 4, 4\sin(t)]$ is smaller or equal than $1/2$ everywhere.

Solution:

Yes, for an ellipse it is largest $1/2$.

9) **T** F It is possible to intersect a cone with a sphere to get an ellipse.

Solution:

One usually gets even a circle

10) **T** F The set of points in space which satisfy $x^2 - y^2 - 1 = z^2$ is a two sheeted hyperboloid.

Solution:

Yes it is

11) **T** F There are examples such that the length of the projection of $\vec{v}$ onto $\vec{w}$ is larger than $|\vec{w}|$.

Solution:

If you project a large vector you get back a large vector

12) T F If $\vec{v}(t), \vec{w}(t)$ are curves, then the derivative $(\vec{v} \times \vec{w})'$ is $\vec{v}' \times \vec{w} + \vec{v} \times \vec{w}'$.

Solution:

This is the product rule.

13) T F The set of points given in spherical coordinates as $\rho^2 \sin^2(\phi) - \rho^2 \cos^2(\phi) = 1$ is a one sheeted hyperboloid.

Solution:

It is one sheeted.

14) T F If A, B, C are three points space and $\vec{AB} \times \vec{AC} = \vec{0}$ then A, B, C are in the same line.

Solution:

Indeed, the area of the parallelepiped is then zero.

15) T F The line $\vec{r}(t) = [-4t, 3t, 0]$ hits the plane $-4x + 3y = 10$ at a right angle.

Solution:

The line is actually parallel to the plane

16) T F The curve given in polar coordinates as $r = \sin(\theta) + 1/r$ is a circle.

Solution:

It translates to $r^2 = r\cos(\theta) + 1$ which is $r^2 = y + 1$ or $x^2 + y^2 = y + 1$.

17) T F If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (3, \pi/2, \pi/2)$, then its rectangular coordinates are $(x, y, z) = (0, 3, 0)$.

Solution:

It has to be on the y-axes.

18) T F The point $(0, -1)$ in $\mathbb{R}^2$ has the polar coordinates $(r, \theta) = (1, 3\pi/2)$.

Solution:

Just check that $r\cos(\theta) = 0$ and $r\sin(\theta) = -1$.

19) T F The surface given in spherical coordinates as $(\rho - 1)(\rho - 2) = 0$ is a union of two spheres.

Solution:

Indeed, either $\rho = 1$ or $\rho = 2$.

20) T F The Möbius strip you have plotted in the homework has the property that its boundary rim consists of two separate closed curves.

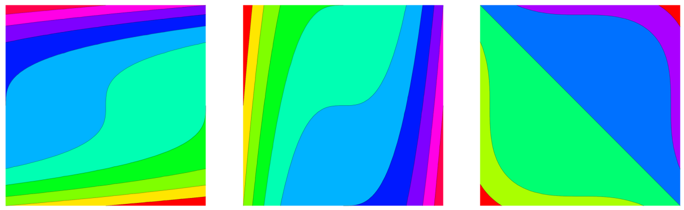
Solution:

The boundary of the Moebius strip has only one rim. It is a single closed curve.

Problem 2) (10 points) No justifications. 0,1,2,3 appear once in a),b),c),d),e)

a) (2 points) Match functions g with their xy-contour plots. Enter 0 if there is no match.

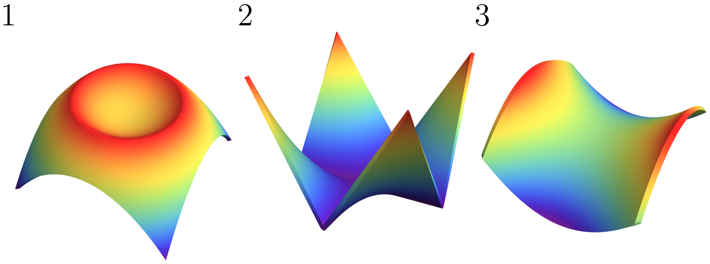
1 2 3



Function $g(x,y) =$	0,1,2, or 3
$x^3 - y^3$	
$x^3 - y$	
$x - y^3$	
$x^3 + y^3$	

b) (2 points) Match the graphs of the functions $f(x,y)$. Enter 0 if there is no match.

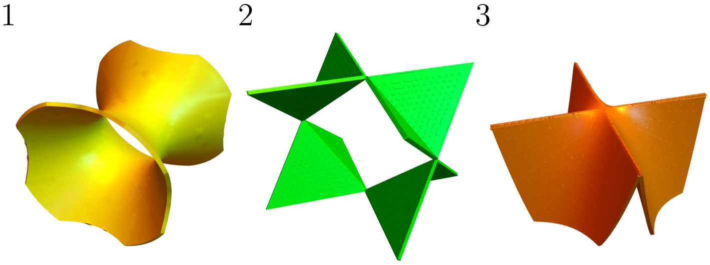
1 2 3



Function $f(x,y) =$	0,1,2, or 3
$\|x\|\|y\|$	
$\|x + y\|$	
$x^2 - y^2$	
$-\|x^2 + y^2 - 1\|$	

c) (2 points) Match the surfaces $g(x,y,z) = c$. Enter 0 if there is no match.

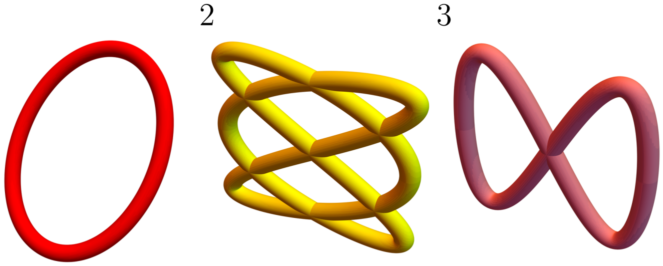
1 2 3



Function $g(x,y,z) =$	0,1,2, or 3
$\|x\| - \|y\| + \|z\| = 1$	
$\|x^2 - y^2\| + z = 1$	
$x^2 - y^2 + z^2 = 1$	
$x^2 - y^2 = 1$	

d) (2 points) Match the space curves with the parametrizations. Enter 0 if there is no match.

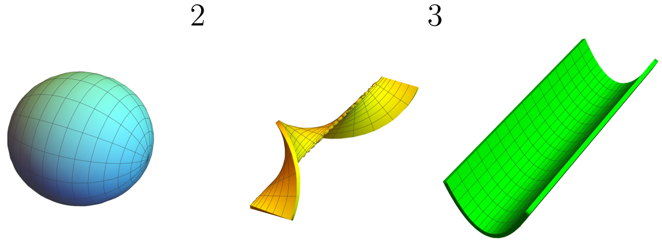
1 2 3



Parametrization $\vec{r}(t) =$	0,1,2, or 3
$[\cos(t), 0, \sin(2t)]$	
$[\cos(t), \sin(t), \sin(t)]$	
$[\sin(t), t^2 - 1, \cos(t)]$	
$[\sin(3t), 0, \sin(2t)]$	

e) (2 points) Match the parametrized surfaces. Enter 0 if there is no match.

1 2 3



Parametrization $\vec{r}(u,v) =$	0-3
$[u, v, u^2]$	
$[u^2\cos(v), v, u^2\sin(v)]$	
$[\cos(v), \sin(v)\cos(u), \sin(v)\sin(u)]$	
$[u, u^2, v]$	

Solution:

a) 0213 b) 2031 c) 2310 d) 3102 e) 3210

Problem 3) (10 points) Each subproblem is 0 or 2 points

Remember that $\vec{i} = [1, 0, 0]$ and $\vec{j} = [0, 1, 0]$ and $\vec{k} = [0, 0, 1]$.

a) (2 points) What is the inner product of $\vec{i}$ with the cross product $\vec{i} \times \vec{j}$.

Answer:

Solution:

Chat GPT: The cross product of two parallel vectors is always zero. Here, both vectors involved are unit vectors in the direction of the x-axis (i.e., they are parallel). So the cross product of $\vec{i}$ and $\vec{i}$ is zero. So, when you then take the inner product of $\vec{i}$ and this cross product (which is zero), you also get zero. Mathematically, we can express this as follows: The cross product of $\vec{i}$ and $\vec{i}$ is: $\vec{i} \times \vec{i} = \vec{0}$ Then the inner product of $\vec{i}$ and this cross product is: $\vec{i} \cdot (\vec{i} \times \vec{i}) = \vec{i} \cdot \vec{0} = 0$

b) (2 points) What is the angle between $\vec{i}$ and $\vec{i} + \vec{j}$.

Answer:

Solution:

Chat GPT Their dot product $\vec{a} \cdot \vec{b}$ is $(\vec{i}) \cdot (\vec{i} + \vec{j}) = 1 \cdot 1 + 0 \cdot 1 = 1$. The magnitudes of these vectors are $||\vec{i}|| = 1$ and $||\vec{i} + \vec{j}|| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Substituting these values into the dot product formula gives: $\cos(\theta) = 1/2$ and so $\theta = \pi/4$.

c) (2 points) Find the cross product of $\vec{i}$ with the cross product $\vec{i} \times \vec{j}$.

Answer:

Solution:

Chat GPT: First, we calculate the cross product of $\vec{i}$ and $\vec{j}$, which is a vector $\vec{k}$ pointing in the direction of the z-axis. Mathematically, we express this as: $\vec{i} \times \vec{j} = \vec{k}$ Then, we take the cross product of $\vec{i}$ and $\vec{k}$, which results in a vector $\vec{j}$ pointing in the direction of the y-axis: $\vec{i} \times \vec{k} = -\vec{j}$ So, the cross product of $\vec{i}$ with the cross product of $\vec{i}$ and $\vec{j}$ is $-\vec{j}$.

d) (2 points) Define $T(\vec{v}) = \vec{i} \times \vec{v}$. Find $T(T(T(T(\vec{j}))))$.

Answer:

Solution:

Chat GPT: The transformation $T(\vec{v})$ applies the cross product of $\vec{i}$ with a vector $\vec{v}$. We have four applications of the transformation, so we'll proceed step by step. First, we find $T(\vec{j})$: $T(\vec{j}) = \vec{i} \times \vec{j} = \vec{k}$ Next, we find $T(T(\vec{j}))$: $T(T(\vec{j})) = T(\vec{k}) = \vec{i} \times \vec{k} = -\vec{j}$. Then we find $T(T(T(\vec{j})))$: $T(T(T(\vec{j}))) = T(-\vec{j}) = \vec{i} \times (-\vec{j}) = -\vec{k}$ Finally, we find $T(T(T(T(\vec{j}))))$: $T(T(T(T(\vec{j})))) = T(-\vec{k}) = \vec{i} \times (-\vec{k}) = \vec{j}$ So, $T(T(T(T(\vec{j})))) = \vec{j}$.

e) (2 points) Define $T(\vec{v}) = \vec{i} \times \vec{v}$ and $T^n(\vec{v}) = T(T^{n-1}(\vec{v}))$. Find $T^{1000}(\vec{j})$.

Answer:

Solution:

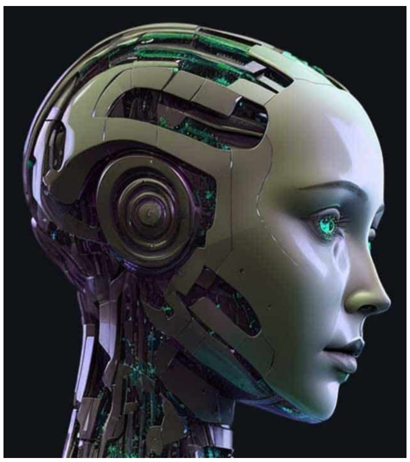
Chat GPT: From your previous question, we can observe a cycle when the transformation T is applied consecutively: $T(\vec{j}) = \vec{k}$ $T(T(\vec{j})) = -\vec{j}$ $T(T(T(\vec{j}))) = -\vec{k}$ $T(T(T(T(\vec{j})))) = \vec{j}$ So we have a period of 4, after which the sequence of transformations repeats. Hence, the result of applying the transformation T any multiple of 4 times will just be the vector we started with. For $n = 1000$, since 1000 is a multiple of 4, we have: $T^{1000}(\vec{j}) = \vec{j}$.

Chat GPT solved all these problems well. We will comment on this after the exam. Here is a bonus problem. If you can solve this correctly (which is not expected), you can regain 2 points (the maximal score can still only be 10 in this problem). In any way, it is a problem which Oliver thought, would confuse Chat GPT because of different contexts. It did not.

f) What is $(|\vec{i}|i)^{(|\vec{i}|i)}$? Answer:

Solution:

Chat GPT: The term $|\vec{i}|$ is the magnitude of the vector $\vec{i}$. Since $\vec{i}$ is the unit vector in the x-direction, its magnitude is 1, so $|\vec{i}| = 1$. The term $i^{|\vec{i}|i}$ is a bit more complex. In this context, it seems you're using i to represent both the imaginary unit (the square root of -1 in complex numbers) and the unit vector in the x-direction. This ambiguity makes the term confusing. If you are using i to represent only the imaginary unit, then i^i is a well-defined complex number. Euler's formula, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, shows that $i = e^{i\pi/2}$. Thus, $i^i = (e^{i\pi/2})^i = e^{-\pi/2}$, a real number. Your expression is equal to $e^{-\pi/2}$.



Declaration: The picture to the right showing an AI taking an exam was AI generated.

Problem 4) (10 points)

a) (5 points) Find the equation of the plane that contains the x axis and the point $(3, 4, 5)$.

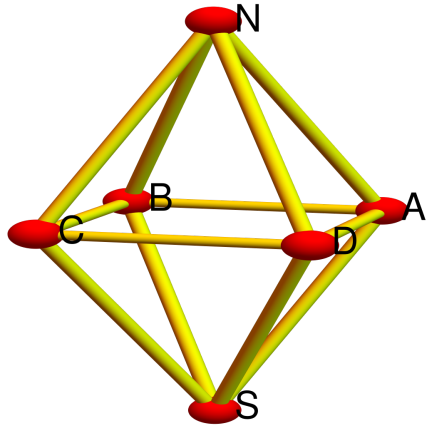
b) (5 points) Parametrize the plane you found in a).

Solution:

Pick the point $(0,0,0)$ and the vector $[1,0,0]$ in the plane. Take the cross product of $[1,0,0]$ with $[3,4,5]$ to get a normal vector $[a,b,c]$ and to write down $ax+by+cz=d$, then plug in the point $(0,0,0)$ to see the constant d. The equation is $-5y+4z=0$. The parametrization is $\vec{r}(t)=[t+3s,4s,5s]$.
We also asked Chat GPT: a) A plane in 3D space can be defined by a point and a normal vector to the plane. Here, the plane we want contains the x-axis and the point (3,4,5). Since the plane contains the x-axis, any point on the x-axis (like the origin, or (1,0,0)) is also in the plane. Let's use the origin and the point (1,0,0) along with the point (3,4,5) to form two vectors that lie in the plane: Vector from the origin to the point (3,4,5): $\vec{v}=(3,4,5)-(0,0,0)=(3,4,5)$ Vector from the origin to the point (1,0,0) on the x-axis: $\vec{u}=(1,0,0)-(0,0,0)=(1,0,0)$ The cross product of these two vectors will give a vector normal (perpendicular) to the plane: $\vec{n}=\vec{u}\times\vec{v}=(0,5,-4)$ We then use this normal vector and any point on the plane to find the equation of the plane. The general equation for a plane is $\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$, where $\vec{n}$ is the normal vector, $\vec{r}$ is a general point on the plane, and $\vec{r}_0$ is a known point on the plane. Substituting the values, we get: $(0,5,-4)\cdot(x-3,y-4,z-5)=0$ which simplifies to the equation of the plane: $5y-4z-20=0$.
b) Chat GPT: We know that the plane contains the x-axis and the point (3,4,5). The x-axis is described by all points of the form (t,0,0), where t is a parameter that can take on any real value. A plane in 3D space can be parametrized using two direction vectors that lie in the plane, and a specific point in the plane. We can take the vectors along the x-axis and a vector pointing from any point on the x-axis to the point (3,4,5) as our direction vectors. Let's take (1,0,0) and (3,4,5)-(1,0,0)=(2,4,5) as the direction vectors. We can take (1,0,0) as a specific point on the plane. Then, the plane can be parametrized as follows: $\vec{r}(s,t)=(1,0,0)+s(1,0,0)+t(2,4,5)$ which simplifies to: $\vec{r}(s,t)=(1+s+2t,4t,5t)$ for all real values of parameters s and t. This describes all points in the plane.

Problem 5) (10 points)

The **octahedron** is polyhedron with 6 vertices, 12 edges and 8 faces. The points $A=(2,2,0), B=(-2,2,0), C=(-2,-2,0), D=(2,-2,0)$ form the equator. The north pole is the point $N=(0,0,2\sqrt{2})$, the south pole is the point $S=(0,0,-2\sqrt{2})$. What is the distance between the line through A,B and the line through C,S?



Solution:

The cross product of $\vec{AB}$ with $\vec{CS}$ is $\vec{n} = [0, -8\sqrt{2}, -8]$. The distance is $|\vec{AC} \cdot \vec{n}|/|\vec{n} = 8\sqrt{2}/\sqrt{3}$

Problem 6) (10 points)

The Harvard Scientist Avi Loeb just concluded a mission in Papa New Guinea. The goal was to find some debris of an interstellar object that crashed into the ocean in 2014. This object CNEOS1 2014-01-08 is the first of this kind which crashed into the earth. It had moved on a hyperbolic path

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \\ 2t \end{bmatrix} .$$

a) (3 points) Verify that $x^2 - y^2 = 4$, confirming so that the path is on a hyperbola.

b) (3 points) Find the curvature at $t = 0$.

c) (4 points) Compute the **arc length** of this curve from $t = -1$ to $t = 1$.

Avi Loeb to the left during the expedition. The right picture was AI generated.

Solution:

a) just foil out both terms and simplify to see that it works.
b) Use the cross product formula $\kappa = 1/2$. c) $\int_{-1}^{1} |\vec{r}\,'(t)|\, dt$ (gave full credit because the 2t term had been missing) The numerical result with the 2t term is $\sqrt{2}(e - 1/e)$.

Problem 7) (10 points)

In this problem we assume $\vec{r}\,'(t) = \begin{bmatrix} \cos(2t) \\ \sin(2t) \\ 0 \end{bmatrix}$.

a) (2 points) Find $\vec{r}(t)$ with $\vec{r}(0) = \vec{j}$.

b) (2 points) What is $\vec{r}\,''(0)$?

c) (2 points) Find the unit tangent vector $\vec{T}(0)$ for $t = 0$.

d) (2 points) Find the normal vector $\vec{N}(0)$ for $t = 0$.

e) (2 points) What is the curvature of the curve at $t = 0$?

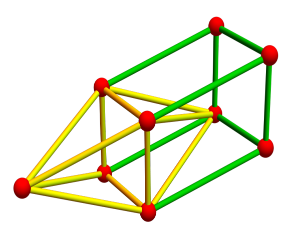
Solution:

a) $[\sin(2t)/2, -\cos(2t)/2 + 3/2, 0]$
b) $[0, 2, 0]$
c) $[1, 0, 0]$
d) $[0, -1, 0]$
e) 2

Problem 8) (10 points)

We look again at the **octahedron** defined by the points $A = (2, 2, 0), B = (-2, 2, 0), C = (-2, -2, 0), D = (2, -2, 0)$, $N = (0, 0, 2\sqrt{2})$ and $S = (0, 0, -2\sqrt{2})$. Compute the volume of the parallelepiped spanned by $\vec{AS}, \vec{AB}, \vec{AD}$ using cross product and dot product.

P.S. The volume of the parallel epiped is also 4/6'th of the volume of the octahedron because the octahedron is made of 4 tetrahedra of volume one sixth of the parallel epiped. We want you however to compute the volume using the triple scalar product.



Solution:

Just compute the triple scalar product. $32\sqrt{2}$. The Chat GPT solution was wrong.

Problem 9) (10 points) No justifications are needed.

Avi Loeb also gained some fame by claiming that the interstellar object **Oumuamua** could be a probe of an alian civilization. We imagination (without AI, we are proud humans after all!) on how such an interstellar space ship might have would looked like. In each problem, please use the provided variables. θ, ϕ have their usual meaning like but possibly in one of the other coordinate plane. The angle ψ was used already in the torus parametrization of the homework.

a) (2 points) A structural **plane** $z = 1$ has the parametrization

$\vec{r}(s,t) = [\quad , \quad , \quad]$

b) (2 points) There is a **torus** which when rotated allows artificial gravity. The angle θ is the polar angle in the yz-plane. The angle ψ is as in the homework problem on the torus. We give you the distance $r = (2 + \cos(\psi))$ to the x-axes and $x = \sin(\psi)$ similarly as in the homework math candy problem. You have to complete the rest:

$\vec{r}(\theta,\psi) = [\sin(\psi), (2+\cos(\psi)) \quad , (2+\cos(\psi)) \quad]$

c) (2 points) There is an ellipsoid $9(x+5)^2 + y^2 + (z-1)^2 = 1$ containing the bulk engine.

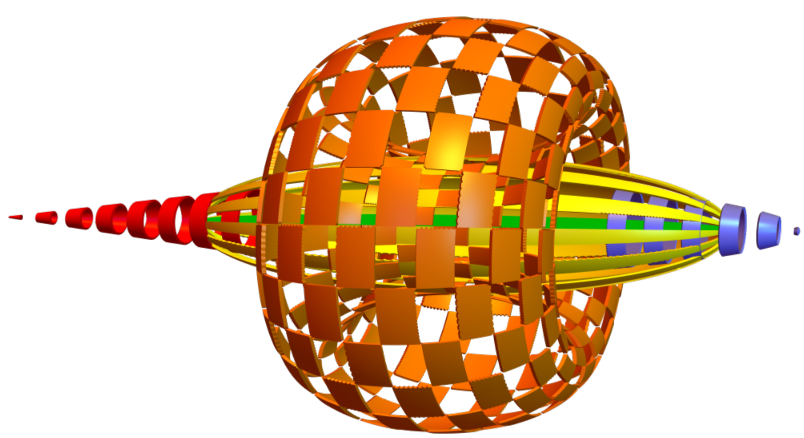
$\vec{r}(\theta,\phi) = [\quad , \quad , \quad]$

d) (2 points) The tail is a paraboloid $x = y^2 + (z-1)^2$.

$\vec{r}(y,z) = [\quad , \quad , \quad]$

e) (2 points) The with cone shaped command capsule $(x+10)^2 = y^2 + (z-1)^2$.

$\vec{r}(x,\theta) = [\quad , \quad , \quad]$



Solution:

a) $\vec{r}(s,t) = [s, t, 1]$
b) $\vec{r}(\theta,\psi) = [\sin(\psi), (2+\cos(\psi))\cos(\theta), (2+\cos(\psi))\sin(\theta)]$
c) $\vec{r}(\theta,\phi) = [5 + 1/3\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$
d) $\vec{r}(y,z) = [y^2 + (z-1)^2, y, z]$
e) $\vec{r}(x,\theta) = [x, (10+x)\cos(\theta), (10+x)\sin(\theta) + 1]$