7/6/2023 FIRST HOURLY **Maths 21a, O.Knill, Summer 2023**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

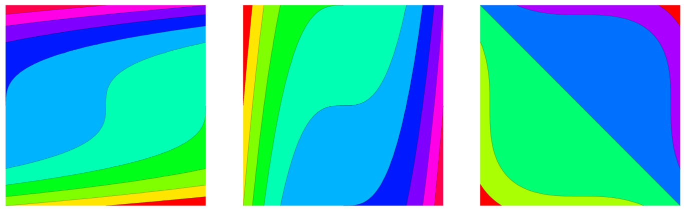
Problem 1) (20 points) No justifications are needed.

1)	T	F	The curvature of $\vec{r}(t)$ is a vector perpendicular to the curve.
2)	T	F	$\vec{v} \times (\vec{w} \times \vec{v}) = \vec{0}$ for all vectors $\vec{v}$ and $\vec{w}$.
3)	T	F	The vector projection $\vec{P}$ is always commutative in the sense $\vec{P}_{\vec{v}}(\vec{P}_{\vec{w}}(\vec{u})) = \vec{P}_{\vec{w}}(\vec{P}_{\vec{v}}(\vec{u}))$.
4)	T	F	If a vector $\vec{v}$ has integer components, then $\vec{v} \cdot \vec{v}$ is an integer.
5)	T	F	The function $f(x, y) = xy/(x^4 + y^4)$ is continuous at $(0, 0)$.
6)	T	F	If $\vec{u}, \vec{v}$ form an acute angle, then $\vec{u}, -\vec{v}$ form an obtuse angle.
7)	T	F	We have $\|\vec{v} \cdot \vec{w}\| \leq \|\vec{v}\|^2$ for all vectors $\vec{v}, \vec{w}$.
8)	T	F	The curvature of the curve $\vec{r}(t) = [2\cos(t), 4, 4\sin(t)]$ is smaller or equal than 1/2 everywhere.
9)	T	F	It is possible to intersect a cone with a sphere to get an ellipse.
10)	T	F	The set of points in space which satisfy $x^2 - y^2 - 1 = z^2$ is a two sheeted hyperboloid.
11)	T	F	There are examples such that the length of the projection of $\vec{v}$ onto $\vec{w}$ is larger than $\|\vec{w}\|$.
12)	T	F	If $\vec{v}(t), \vec{w}(t)$ are curves, then the derivative $(\vec{v} \times \vec{w})'$ is $\vec{v}' \times \vec{w} + \vec{v} \times \vec{w}'$.
13)	T	F	The set of points given in spherical coordinates as $\rho^2 \sin^2(\phi) - \rho^2 \cos^2(\phi) = 1$ is a one sheeted hyperboloid.
14)	T	F	If A, B, C are three points space and $\vec{AB} \times \vec{AC} = \vec{0}$ then A, B, C are in the same line.
15)	T	F	The line $\vec{r}(t) = [-4t, 3t, 0]$ hits the plane $-4x + 3y = 10$ at a right angle.
16)	T	F	The curve given in polar coordinates as $r = \sin(\theta) + 1/r$ is a circle.
17)	T	F	If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (3, \pi/2, \pi/2)$, then its rectangular coordinates are $(x, y, z) = (0, 3, 0)$.
18)	T	F	The point $(0, -1)$ in $\mathbb{R}^2$ has the polar coordinates $(r, \theta) = (1, 3\pi/2)$.
19)	T	F	The surface given in spherical coordinates as $(\rho - 1)(\rho - 2) = 0$ is a union of two spheres.
20)	T	F	The Möbius strip you have plotted in the homework has the property that its boundary rim consists of two separate closed curves.

Problem 2) (10 points) No justifications. 0,1,2,3 appear once in a),b),c),d),e)

a) (2 points) Match functions g with their xy-contour plots. Enter 0 if there is no match.

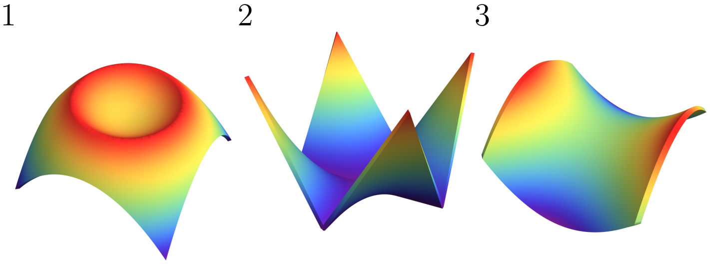
1 2 3



Function $g(x,y) =$	0,1,2, or 3
$x^3 - y^3$	
$x^3 - y$	
$x - y^3$	
$x^3 + y^3$	

b) (2 points) Match the graphs of the functions $f(x,y)$. Enter 0 if there is no match.

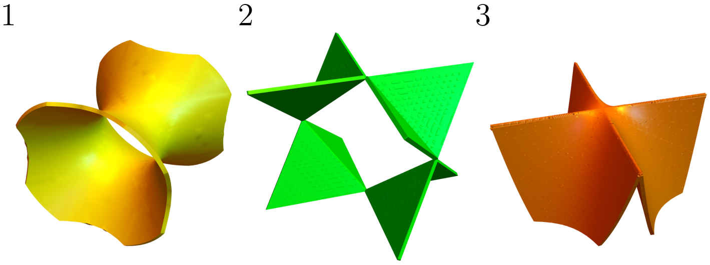
1 2 3



Function $f(x,y) =$	0,1,2, or 3
$\lvert x\rvert\lvert y\rvert$	
$\lvert x+y\rvert$	
$x^2 - y^2$	
$-\lvert x^2+y^2-1\rvert$	

c) (2 points) Match the surfaces $g(x,y,z) = c$. Enter 0 if there is no match.

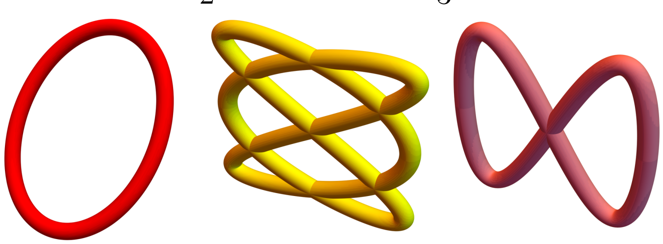
1 2 3



Function $g(x,y,z) =$	0,1,2, or 3
$\lvert x\rvert - \lvert y\rvert + \lvert z\rvert = 1$	
$\lvert x^2 - y^2\rvert + z = 1$	
$x^2 - y^2 + z^2 = 1$	
$x^2 - y^2 = 1$	

d) (2 points) Match the space curves with the parametrizations. Enter 0 if there is no match.

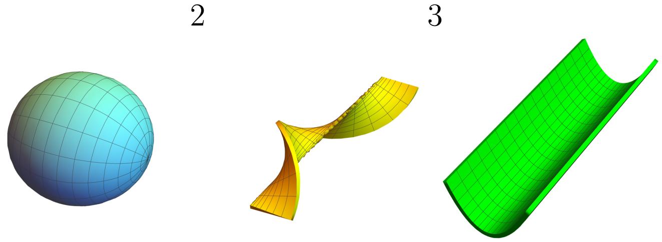
1 2 3



Parametrization $\vec{r}(t) =$	0,1,2, or 3
$[\cos(t), 0, \sin(2t)]$	
$[\cos(t), \sin(t), \sin(t)]$	
$[\sin(t), t^2 - 1, \cos(t)]$	
$[\sin(3t), 0, \sin(2t)]$	

e) (2 points) Match the parametrized surfaces. Enter 0 if there is no match.

1 2 3



Parametrization $\vec{r}(u,v) =$	0-3
$[u, v, u^2]$	
$[u^2\cos(v), v, u^2\sin(v)]$	
$[\cos(v), \sin(v)\cos(u), \sin(v)\sin(u)]$	
$[u, u^2, v]$	

Problem 3) (10 points) Each subproblem is 0 or 2 points

Remember that $\vec{i} = [1, 0, 0]$ and $\vec{j} = [0, 1, 0]$ and $\vec{k} = [0, 0, 1]$.

a) (2 points) What is the inner product of $\vec{i}$ with the cross product $\vec{i} \times \vec{j}$.

Answer:

b) (2 points) What is the angle between $\vec{i}$ and $\vec{i} + \vec{j}$.

Answer:

c) (2 points) Find the cross product of $\vec{i}$ with the cross product $\vec{i} \times \vec{j}$.

Answer:

d) (2 points) Define $T(\vec{v}) = \vec{i} \times \vec{v}$. Find $T(T(T(T(\vec{j}))))$.

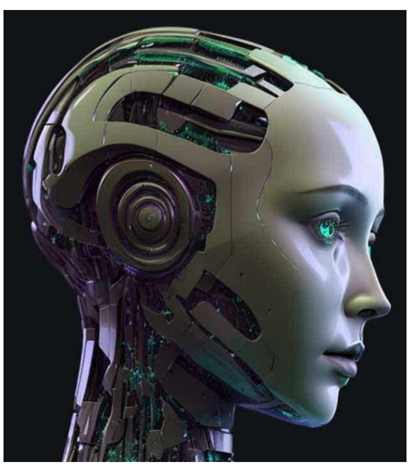
Answer:

e) (2 points) Define $T(\vec{v}) = \vec{i} \times \vec{v}$ and $T^n(\vec{v}) = T(T^{n-1}(\vec{v}))$. Find $T^{1000}(\vec{j})$.

Answer:

Chat GPT solved all these problems well. We will comment on this after the exam. Here is a bonus problem. If you can solve this correctly (which is not expected), you can regain 2 points (the maximal score can still only be 10 in this problem). In any way, it is a problem which Oliver thought, would confuse Chat GPT because of different contexts. It did not.

f) What is $(|\vec{i}|i)^{(|\vec{i}|i)}$? Answer:



Declaration: The picture to the right showing an AI taking an exam was AI generated.

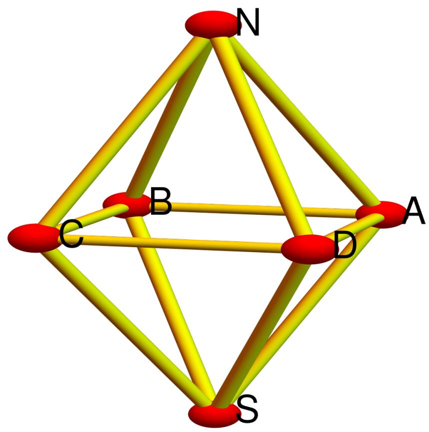
Problem 4) (10 points)

a) (5 points) Find the equation of the plane that contains the x axis and the point $(3, 4, 5)$.

b) (5 points) Parametrize the plane you found in a).

Problem 5) (10 points)

The **octahedron** is polyhedron with 6 vertices, 12 edges and 8 faces. The points $A = (2, 2, 0), B = (-2, 2, 0), C = (-2, -2, 0), D = (2, -2, 0)$ form the equator. The north pole is the point $N = (0, 0, 2\sqrt{2})$, the south pole is the point $S = (0, 0, -2\sqrt{2})$. What is the distance between the line through A, B and the line through C, S?



Problem 6) (10 points)

The Harvard Scientist Avi Loeb just concluded a mission in Papa New Guinea. The goal was to find some debris of an interstellar object that crashed into the ocean in 2014. This object CNEOS1 2014-01-08 is the first of this kind which crashed into the earth. It had moved on a hyperbolic path

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \\ 2t \end{bmatrix} .$$

a) (3 points) Verify that $x^2 - y^2 = 4$, confirming so that the path is on a hyperbola.

b) (3 points) Find the curvature at $t = 0$.

c) (4 points) Compute the **arc length** of this curve from $t = -1$ to $t = 1$.


Avi Loeb to the left during the expedition. The right picture was AI generated.

Problem 7) (10 points)

In this problem we assume $\vec{r}\,'(t) = \begin{bmatrix} \cos(2t) \\ \sin(2t) \\ 0 \end{bmatrix}$.

a) (2 points) Find $\vec{r}(t)$ with $\vec{r}(0) = \vec{j}$.

b) (2 points) What is $\vec{r}\,''(0)$?

c) (2 points) Find the unit tangent vector $\vec{T}(0)$ for $t = 0$.

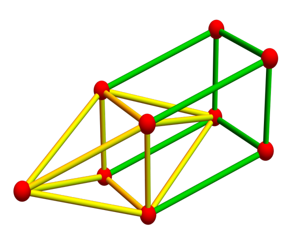
d) (2 points) Find the normal vector $\vec{N}(0)$ for $t = 0$.

e) (2 points) What is the curvature of the curve at $t = 0$?

Problem 8) (10 points)

We look again at the **octahedron** defined by the points $A = (2,2,0), B = (-2,2,0), C = (-2,-2,0), D = (2,-2,0)$, $N = (0,0,2\sqrt{2})$ and $S = (0,0,-2\sqrt{2})$. Compute the volume of the parallelepiped spanned by $\vec{AS}, \vec{AB}, \vec{AD}$ using cross product and dot product.

P.S. The volume of the parallel epiped is also 4/6'th of the volume of the octahedron because the octahedron is made of 4 tetrahedra of volume one sixth of the parallel epiped. We want you however to compute the volume using the triple scalar product.



Problem 9) (10 points) No justifications are needed.

Avi Loeb also gained some fame by claiming that the interstellar object **Oumuamua** could be a probe of an alian civilization. We imagination (without AI, we are proud humans after all!) on how such an interstellar space ship might have would looked like. In each problem, please use the provided variables. θ, ϕ have their usual meaning like but possibly in one of the other coordinate plane. The angle ψ was used already in the torus parametrization of the homework.

a) (2 points) A structural **plane** $z = 1$ has the parametrization

$\vec{r}(s,t) = [\quad , \quad , \quad]$

b) (2 points) There is a **torus** which when rotated allows artificial gravity. The angle θ is the polar angle in the yz-plane. The angle ψ is as in the homework problem on the torus. We give you the distance $r = (2 + \cos(\psi))$ to the x-axes and $x = \sin(\psi)$ similarly as in the homework math candy problem. You have to complete the rest:

$\vec{r}(\theta,\psi) = [\sin(\psi), (2+\cos(\psi)) \quad , (2+\cos(\psi)) \quad]$

c) (2 points) There is an ellipsoid $9(x+5)^2 + y^2 + (z-1)^2 = 1$ containing the bulk engine.

$\vec{r}(\theta,\phi) = [\quad , \quad , \quad]$

d) (2 points) The tail is a paraboloid $x = y^2 + (z-1)^2$.

$\vec{r}(y,z) = [\quad , \quad , \quad]$

e) (2 points) The with cone shaped command capsule $(x+10)^2 = y^2 + (z-1)^2$.

$\vec{r}(x,\theta) = [\quad , \quad , \quad]$