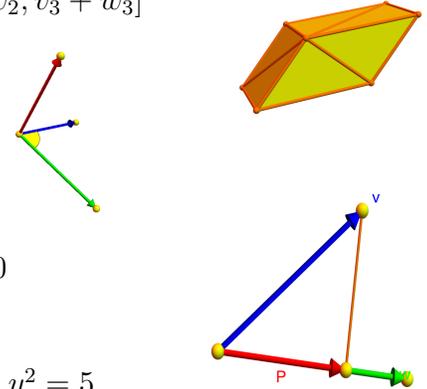


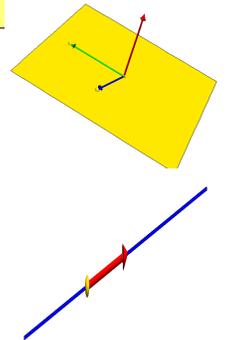
Geometry of Space

- unit vector = direction = vector of length 1.
- $\vec{v} = [v_1, v_2, v_3], \vec{w} = [w_1, w_2, w_3], \vec{v} + \vec{w} = [v_1 + w_1, v_2 + w_2, v_3 + w_3]$
- dot product $\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3 = |\vec{v}||\vec{w}| \cos(\alpha)$
- angle $\cos(\alpha) = (\vec{v} \cdot \vec{w}) / |\vec{v}||\vec{w}|$.
- cross product $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0, \vec{w} \cdot (\vec{v} \times \vec{w}) = 0$
- triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w})$
- area of parallelogram $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$
- volume of parallelepiped: $|\vec{u} \cdot (\vec{v} \times \vec{w})|$
- parallel vectors $\vec{v} \times \vec{w} = 0$, orthogonal vectors: $\vec{v} \cdot \vec{w} = 0$
- scalar projection $\vec{v} \cdot \vec{w} / |\vec{w}|$
- vector projection $P_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \vec{w})\vec{w} / |\vec{w}|^2$
- completion of square: $x^2 - 4x + y^2 = 1$ gives $(x - 2)^2 + y^2 = 5$
- sphere: $(x - 1)^2 + (y - b)^2 + (z - c)^2 = \rho^2$



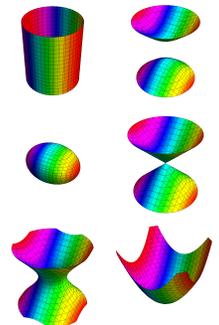
Lines, Planes, Functions

- parametric equation for plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$
- plane $ax + by + cz = d$
- parametric equation for line $\vec{x} = \vec{x}_0 + t\vec{v}$
- symmetric equation of line $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$
- graph $G = \{(x, y, f(x, y)) \mid (x, y) \text{ in the domain of } f\}$
- plane $ax + by + cz = d$ has normal vector $\vec{n} = [a, b, c]$
- plane through A, B, C : us $[a, b, c] = \vec{AB} \times \vec{CB}$ and plug in point



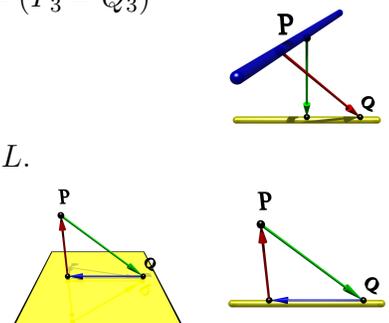
Level surfaces

- intercepts: intersections of a surface with coordinate axis
- traces: intersections of a surface with coordinate planes
- generalized traces: intersections with $\{x = c\}, \{y = c\}$ or $\{z = c\}$
- level surface $g(x, y, z) = c$. Example: graph $g(x, y, z) = z - f(x, y)$
- linear equation like $2x + 3y + 5z = 7$ defines plane
- quadric: ellipsoid, paraboloid, hyperboloid, cylinder, cone



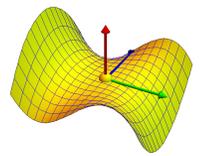
Distance formulas

- point-point: $d(P, Q) = |\vec{PQ}| = \sqrt{(P_1 - Q_1)^2 + (P_2 - Q_2)^2 + (P_3 - Q_3)^2}$
- point-plane: $d(P, \Sigma) = |(\vec{PQ}) \cdot \vec{n}| / |\vec{n}|$
- point-line: $d(P, L) = |(\vec{PQ}) \times \vec{u}| / |\vec{u}|$
- line-line: $d(L, M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})| / |\vec{u} \times \vec{v}|$
- parallel lines: L, M : distance point $d(P, M)$, where P is in L .
- parallel planes: $d(P, \Sigma)$ where P is in first plane.



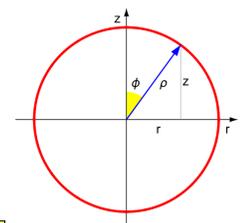
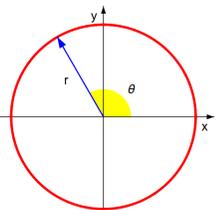
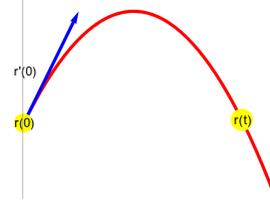
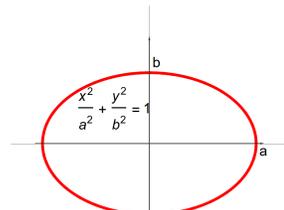
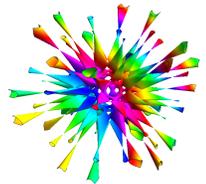
Functions

- graph: $z = f(x, y)$ is a surface in \mathbb{R}^3 .
- contour set: $f(x, y) = c$ is usually a curve in \mathbb{R}^2
- contour map: draw curves $f(x, y) = c$ for various c .
- contour surface: $f(x, y, z) = c$ is usually a surface in space



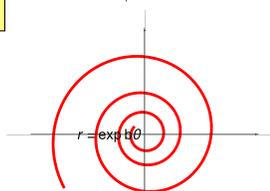
Curves

- plane curve $\vec{r}(t) = [x(t), y(t)]$
- space curve $\vec{r}(t) = [x(t), y(t), z(t)]$
- circle: $x^2 + y^2 = r^2$, $\vec{r}(t) = [r \cos t, r \sin t]$.
- ellipse: $(x - x_0)^2/a^2 + (y - y_0)^2/b^2 = 1$, $\vec{r}(t) = [x_0 + a \cos t, y_0 + b \sin t]$
- velocity $\vec{r}'(t)$, acceleration $\vec{r}''(t)$, $|\vec{r}'(t)|$ speed
- unit tangent vector $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$
- integration: get $\vec{r}(t)$ from $\vec{r}'(t)$ and $\vec{r}(0)$ by integration.
- integration: get $\vec{r}(t)$ from acceleration $\vec{r}''(t)$ as well as $\vec{r}'(0)$ and $\vec{r}(0)$.
- velocity $\vec{r}'(t)$ tangent to curve at $\vec{r}(t)$.
- polar curve $\vec{r}(t) = [f(t) \cos(t), f(t) \sin(t)]$ to polar graph $r = f(\theta)$.
- arc length $\int_a^b |\vec{r}'(t)| dt$ of parametrized curve.
- normal vector $\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$ perpendicular to $\vec{T}(t)$.
- bi-normal $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ perpendicular to \vec{T} and \vec{N} .
- curvature $\kappa(t) = |\vec{T}'(t)|/|\vec{r}'(t)| = |\vec{r}'(t) \times \vec{r}''(t)|/|\vec{r}'(t)|^3$.
- curvature and arc length are independent of parametrization



Coordinates

- Cartesian coordinates (x, y, z)
- polar coordinates $(x, y) = (r \cos(\theta), r \sin(\theta))$, $r \geq 0$
- cylindrical coordinates $(x, y, z) = (r \cos(\theta), r \sin(\theta), z)$, $r \geq 0$
- spherical coordinates $(x, y, z) = (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi))$
- radius: $r = x^2 + y^2$ and spherical radius $\rho = x^2 + y^2 + z^2$.
- polar radius: important relation $r = \rho \sin(\phi)$



Surfaces

- $\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ parametrization
- $\vec{r}(t) = \vec{r}(u_0, t)$, $\vec{r}(t) = \vec{r}(t, v_0)$ grid curves on surface
- $g(r, \theta) = 0$ polar curve, especially $r = f(\theta)$, polar graph
- $r = f(z, \theta)$ cylindrical surface, $r = r(z)$ surface of revolution
- $g(\rho, \theta, \phi) = 0$ spherical surface: example $\rho = 1$ sphere
- plane: $ax + by + cz = d$, $\vec{r}(s, t) = \vec{r}_0 + s\vec{v} + t\vec{w}$, $[a, b, c] = \vec{v} \times \vec{w}$
- surface of revolution: $x^2 + y^2 = r(z)^2$, $\vec{r}(\theta, z) = [r(z) \cos(\theta), r(z) \sin(\theta), z]$
- graph: $g(x, y, z) = z - f(x, y) = 0$, $\vec{r}(x, y) = [x, y, f(x, y)]$
- rotated graph $g(x, y, z) = y - f(x, z) = 0$, $\vec{r}(x, z) = [x, f(x, z), z]$
- ellipsoid: $\vec{r}(\theta, \phi) = [a \cos \theta \sin \phi, b \sin \theta \sin \phi, c \cos \phi]$
- unit sphere: $x^2 + y^2 + z^2 = 1$, $\vec{r}(u, v) = [\cos u \sin v, \sin u \sin v, \cos v]$

