MULTIVARIABLE CALCULUS

MATH S-21A

Unit 23: Stokes Theorem

Lecture

23.1. If a surface S is parametrized as $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$ over a domain R in the uv-plane, the **flux integral** of $\vec{F}$ through S is defined as the double integral

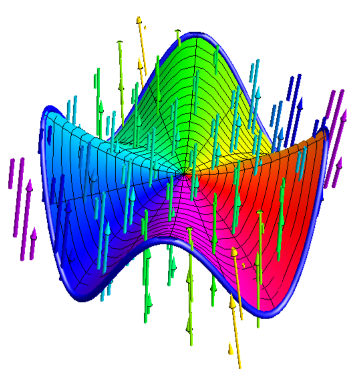
$$\iint_R \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv \, .$$

Definition: The **boundary** of a surface S consists of all points P for which we do not have an entire disc around P which is contained in S. For the upper hemisphere for example, the boundary is the equator, a circle. The boundary of a sphere is empty.

23.2. The boundary of S is a collection of closed curves. Each is oriented so that the surface is to the "left" if the normal vector to the surface is pointing "up". In other words, the velocity vector $\vec{v}$, a vector $\vec{w}$ pointing towards the surface and the normal vector $\vec{n}$ to the surface are right-handed.

Theorem: **Stokes theorem**: if S is a surface bounded by a curve C and $\vec{F}$ be a vector field, then

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \, .$$



23.3. Stokes theorem can be verified in the same way than Green's theorem. Chop up S into a collection of small triangles. As before, the sum of the fluxes through all these triangles adds up to the flux through the surface and the sum of the line integrals along the boundaries adds up to the line integral of the boundary of S. Stokes theorem for a small triangle can be reduced to Green's theorem because with a coordinate system such that the triangle.

Examples

23.4. Let $\vec{F}(x,y,z) = [-y,x,0]$ and let S be the upper hemisphere oriented upwards. We have $\mathrm{curl}(\vec{F})(x,y,z) = [0,0,2]$. The surface is parameterized by

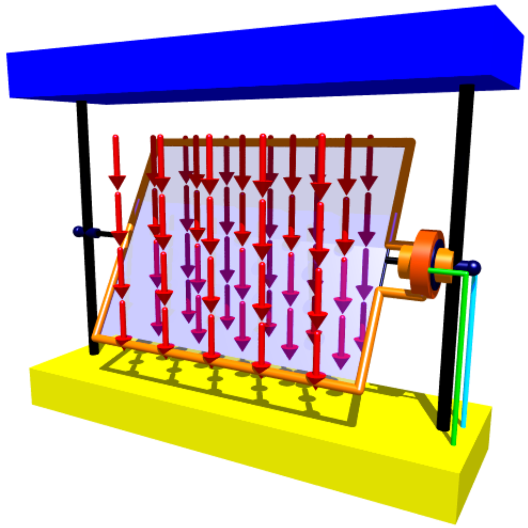
$$\vec{r}(\phi,\theta) = [\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)]$$

on $R = [0,\pi/2] \times [0,2\pi]$ and $\vec{r}_\phi \times \vec{r}_\theta = \sin(\phi)\vec{r}(\phi,\theta)$ so that $\mathrm{curl}(\vec{F})(x,y,z) \cdot \vec{r}_\phi \times \vec{r}_\theta = \cos(\phi)\sin(\phi)2$. The integral $\int_0^{\pi/2}\int_0^{2\pi} \sin(2\phi)\ d\theta d\phi = 2\pi$.
The boundary C of S is parameterized by $\vec{r}(t) = [\cos(t),\sin(t),0]$ so that $\vec{dr} = \vec{r}\,'(t)dt = [-\sin(t),\cos(t),0]\ dt$ and $\int \vec{F}(\vec{r}(t))\ \vec{r}\,'(t)dt = \int_0^{2\pi} \sin(t)^2 + \cos^2(t)\ dt = 2\pi$.

23.5. If S is a surface in the xy-plane and $\vec{F} = [P,Q,0]$ has a zero z-component, then $\mathrm{curl}(\vec{F}) = [0,0,Q_x - P_y]$ and $\mathrm{curl}(\vec{F}) \cdot \vec{dS} = Q_x - P_y\ dxdy$. We see that for a surface which is flat, Stokes theorem is very close to Green's theorem. If we put the coordinate axis so that the surface is in the xy-plane, then the vector field $\vec{F}$ induces a vector field on the surface such that its $2D$-curl is the normal component of $\mathrm{curl}(F)$. The third component $Q_x - P_y$ of $\mathrm{curl}(\vec{F})[R_y - Q_z, P_z - R_x, Q_x - P_y]$ is the two-dimensional $\mathrm{curl}(\vec{F})(\vec{r}(u,v)) \cdot [0,0,1] = Q_x - P_y$. If C is the boundary of the surface, then $\int\int_S \vec{F}(\vec{r}(u,v)) \cdot [0,0,1]\ dudv = \int_C \vec{F}(\vec{r}(t))\ \vec{r}\,'(t)dt$.

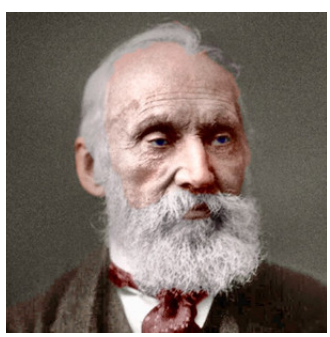
23.6. For every surface bounded by a curve C, the flux of $\mathrm{curl}(\vec{F})$ through the surface is the same. **Proof:** the flux of the curl of a vector field through a surface S depends only on the boundary of S. Compare this with the earlier statement that for every curve between two points A, B the line integral of $\mathrm{grad}(f)$ along C is the same. The line integral of the gradient of a function of a curve C depends only on the end points of C.

23.7. Electric and magnetic fields are linked by the **Maxwell equation** like $\mathrm{curl}(\vec{E}) = -\frac{1}{c}\dot{B}$, an example of a partial differential equation. If a closed wire C bounds a surface S, then $\int\int_S B \cdot dS$ is the flux of the magnetic field through S. Its change can be related with a voltage using Stokes theorem: $d/dt \int\int_S B \cdot dS = \int\int_S \dot{B} \cdot dS = \int\int_S -c\ \mathrm{curl}(\vec{E}) \cdot \vec{dS} = -c\int_C \vec{E}\ \vec{dr} = U$, where U is the voltage. If we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire or the wire inside the magnet, we get an electric voltage. This happens in a power-generator, like the alternator in a car. Stokes theorem explains why we can generate electricity from motion.



23.8. The history of Stokes theorem is a bit hazy. [1]. A version of Stokes theorem was known by **André Ampère** in 1825. **William Thomson** (Lord Kelvin) mentioned the theorem to Stokes in 1850). **George Gabriel Stokes** (1819-1903), who found parts of the identity earlier 1840, formulated it in a prize exam from 1854 in which giving a proof was one of the exam problems. The first published proof of Stokes theorem was provided by Hermann Hankel in 1861.

George Gabriel Stokes



William Thomson (Kelvin)

André Marie Ampere

[1]See V. Katz, the History of Stokes theorem, Mathematics Magazine 52, 1979, p 146-156

Homework

This homework is due on Tuesday, 8/1/2023.

Problem 23.1: Find $\int_C \vec{F} \cdot \vec{dr}$, where $\vec{F}(x,y,z) = [3x^2y, x^3, 3xy]$ and C is the curve obtained by intersecting the hyperbolic paraboloid $z = y^2 - x^2$ with the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

Problem 23.2: Assume S is the surface $x^{1000} + y^{8000} + z^{1000} = 1000000$ and $\vec{F} = [9+e^{e^{xyz}}, x^8yz, 3x-5y-6\cos(zx)]$. Explain why $\iint_S \text{curl}(\vec{F}) \cdot \vec{dS} = 0$.

Problem 23.3: Evaluate the flux integral $\int\int_S \text{curl}(\vec{F}) \cdot \vec{dS}$, where

$$\vec{F}(x,y,z) = [xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x]$$

and where S is the part of the ellipsoid $x^2 + y^2/4 + (z+1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

Problem 23.4: Find the line integral $\int_C \vec{F}\ \vec{dr}$, where C is the circle of radius 5 in the xz-plane oriented counter clockwise when looking from the point $(0,1,0)$ onto the plane and where $\vec{F}$ is the vector field

$$\vec{F}(x,y,z) = [x^2z + x^5, \cos(e^y), -xz^2 + \sin(\sin(z)] .$$

Use a convenient surface S which has C as a boundary.

Problem 23.5: Find the flux integral $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where $\vec{F}(x,y,z) =$

$$[-y + 2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi\sin(\pi y)e^{2x}, 2xz]$$

and S is the surface parametrized by

$$\vec{r}(s,t) = [(1-s^{1/3})\cos(t) - 4s^2, (1-s^{1/3})\sin(t), 5s]$$

with $0 \leq t \leq 2\pi, 0 \leq s \leq 1$ and oriented so that the normal vectors point to the outside of the thorn.

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