MULTIVARIABLE CALCULUS

MATH S-21A

Unit 20: Line integral theorem

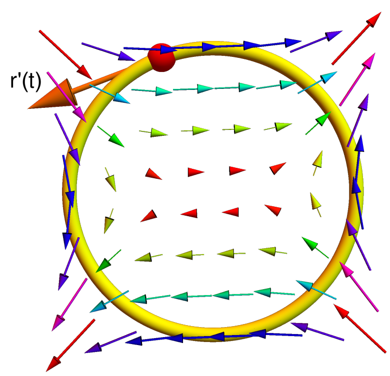
Lecture

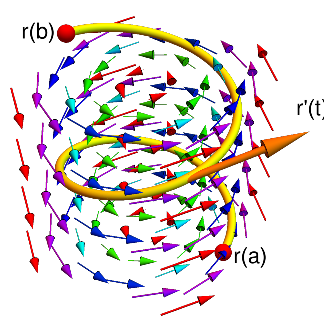
20.1. When a vector field is integrated along a curve, we get a **line integral**. In the special case, where $\vec{F}$ is a gradient field, we can compute the integral using the fundamental theorem of calculus. The corresponding formula will is the first generalization of the fundamental theorem to higher dimensions.

Definition: If $\vec{F}$ is a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $C : t \mapsto \vec{r}(t)$ is a curve, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$$

is called the **line integral** of $\vec{F}$ along the curve C.





20.2. We use also the short-hand notation $\int_C \vec{F} \cdot \vec{dr}$, where $\vec{dr} = \vec{r}\,'(t)dt$ is an abbreviation. In physics, if $\vec{F}(x, y, z)$ is a **force field**, then $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is called **power** and the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ is **work**. In electrodynamics, if $\vec{F}(x, y, z)$ is an electric field, then the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ is the **electrostatic potential**.

20.3. Let $C : t \mapsto \vec{r}(t) = [\cos(t), \sin(t)]$ be a circle parameterized by $t \in [0, 2\pi]$ and let $\vec{F}(x, y) = [-y, x]$. Calculate the line integral $I = \int_C \vec{F}(\vec{r}) \cdot \vec{dr}$.
Solution: We have $I = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\; dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\; dt = 2\pi$

20.4. Let $\vec{r}(t)$ be a curve given in polar coordinates as $\vec{r}(t) = [r(t), \phi(t)] = [\cos(t), t]$ defined on the interval $0 \leq t \leq \pi$. Let $\vec{F}$ be the vector field $\vec{F}(x, y) = [-xy, 0]$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$.
Solution: In Cartesian coordinates, the curve is $\vec{r}(t) = [\cos^2(t), \cos(t)\sin(t)]$. The velocity vector is then $\vec{r}\,'(t) = [-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t)]$. The line integral is

$$\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_0^\pi [\cos^3(t)\sin(t), 0] \cdot [-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)]\; dt$$

$$= \; -2\int_0^\pi \sin^2(t)\cos^4(t)\; dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8 \; .$$

20.5. The first generalization of the fundamental theorem of calculus to higher dimensions is the **fundamental theorem of line integrals**.

Theorem: **Fundamental theorem of line integrals**: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = f(\vec{r}(b)) - f(\vec{r}(a)) \; .$$

20.6. In other words, the line integral is the potential difference between the end points $\vec{r}(b)$ and $\vec{r}(a)$, if $\vec{F}$ is a gradient field. For the proof (your turn) just mind the **chain rule** and the **fundamental theorem of calculus**.

Examples

20.7. Let $f(x, y, z)$ be the temperature distribution in a room and let $\vec{r}(t)$ the path of a fly in the room, then $f(\vec{r}(t))$ is the temperature, the fly experiences at the point $\vec{r}(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt} f(\vec{r}(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference between the end point and initial point.

20.8. Here are some special cases: If $\vec{r}(t)$ is parallel to the level curve of f, then $d/dt f(\vec{r}(t)) = 0$ because $\vec{r}\,'(t)$ is orthogonal to $\nabla f(\vec{r}(t))$. If $\vec{r}(t)$ is orthogonal to the level curve, then $|d/dt f(\vec{r}(t))| = |\nabla f||\vec{r}\,'(t)|$ because $\vec{r}\,'(t)$ is parallel to $\nabla f(\vec{r}(t))$.

20.9. The proof of the fundamental theorem uses the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\; dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\; dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\; dt = f(\vec{r}(b)) - f(\vec{r}(a)) \; .$$

Theorem: For a gradient field, the line-integral along any closed curve is zero.

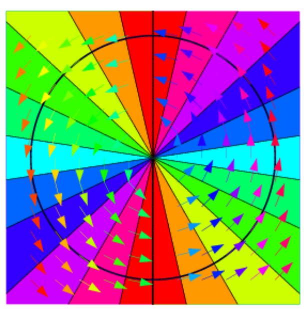
20.10. When is a vector field a gradient field? $\vec{F}(x, y) = \nabla f(x, y)$ implies $P_y(x, y) = Q_x(x, y)$. If this does not hold at some point, $\vec{F} = [P, Q]$ is no gradient field. This is called the **Clairaut test**. We will see later that the condition $\text{curl}(\vec{F}) = Q_x - P_y = 0$ and $\vec{F}$ being defined everywhere implies that the field is a gradient field

20.11. Let $\vec{F}(x,y) = [2xy^2 + 3x^2, 2yx^2]$. Find a potential f of $\vec{F} = [P,Q]$.
Solution: The potential function $f(x,y)$ satisfies $f_x(x,y) = 2xy^2 + 3x^2$ and $f_y(x,y) = 2yx^2$. Integrating the second equation gives $f(x,y) = x^2y^2 + h(x)$. Partial differentiation with respect to x gives $f_x(x,y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x,y) = x^2y^2 + x^3$. Find g,h from $f(x,y) = \int_0^x P(x,y)\,dx + h(y)$ and $f_y(x,y) = g(x,y)$.

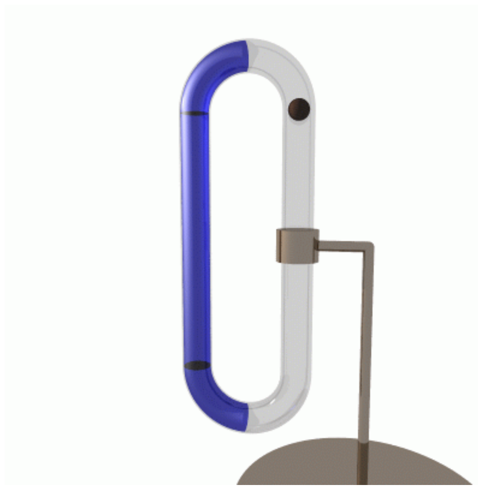
20.12. Let $\vec{F}(x,y) = [P,Q] = [\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}]$. It appears to be a gradient field because $f(x,y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1+y^2/x^2) = P, f_y = (1/x)/(1+y^2/x^2) = Q$. However, the line integral $\int_\gamma \vec{F}\,\vec{dr}$, where γ is the unit circle is

$$\int_0^{2\pi} [\frac{-\sin(t)}{\cos^2(t)+\sin^2(t)}, \frac{\cos(t)}{\cos^2(t)+\sin^2}] \cdot [-\sin(t), \cos(t)]\,dt$$

which is $\int_0^{2\pi} 1\,dt = 2\pi$. What is wrong?
Solution: note that the potential f as well as the vector-field $\vec{F}$ are not differentiable everywhere. The curl of $\vec{F}$ is zero except at $(0,0)$, where it is not defined.



20.13. A device which implements a non gradient force field is called a **perpetual motion machine**. It realizes a force field for which the energy gain is positive along some closed loop. The first law of thermodynamics forbids the existence of such a machine. It is informative to contemplate some of the ideas people have come up and to analyze why they don't work. Here is an example: consider a O-shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water. You find plenty of other futile attempts on youtube. The authors of such videos are no idiots, they rely on a gullible audience for clicks.



Homework

This homework is due on Tuesday, 7/25/2023.

Problem 20.1: What is the work done by moving in the force field $\vec{F}(x,y) = [6x^2+2, 16y^7]$ along the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$? In part a) compute it directly. Then, in part b), use the theorem.

Problem 20.2: Let C be the space curve $\vec{r}(t) = [\cos(t), \sin(\sin(t)), 5t]$ for $t \in [0,\pi]$ and let $\vec{F}(x,y,z) = [y, x, 15+\cos(21z)]$. Find the value of the line integral $\int_C \vec{F} \cdot \vec{dr}$.

Problem 20.3: Let $\vec{F}$ be the vector field $\vec{F}(x,y) = [-y,x]/2$. Compute the line integral of $\vec{F}$ along an ellipse $\vec{r}(t) = [a\cos(t), b\sin(t)]$ with width $2a$ and height $2b$. The result should depend on a and b.

Problem 20.4: It is hot and you refresh yourself in a little pool in your garden. Its rim has the shape $x^{40}+y^{40} = 1$ oriented counter clockwise. There is a hose filling in fresh water to the tub so that there is a velocity field $\vec{F}(x,y) = [2x+5y, 10y^4+5x]$ inside. Calculate the line integral $\int_C \vec{F} \cdot \vec{dr}$, the energy you gain from the fluid force when dislocating from $(1,0)$ to $(0,1)$ along the rim. Remember you are in a pool and do not want to work hard. There is an easy way to get the answer.

Problem 20.5: Find a closed curve $C : \vec{r}(t)$ for which the vector field

$$\vec{F}(x,y) = [P(x,y), Q(x,y)] = [xy, x^2]$$

satisfies $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \neq 0$.