

# MULTIVARIABLE CALCULUS

MATH S-21A

## Unit 16: Surface Integration

### LECTURE

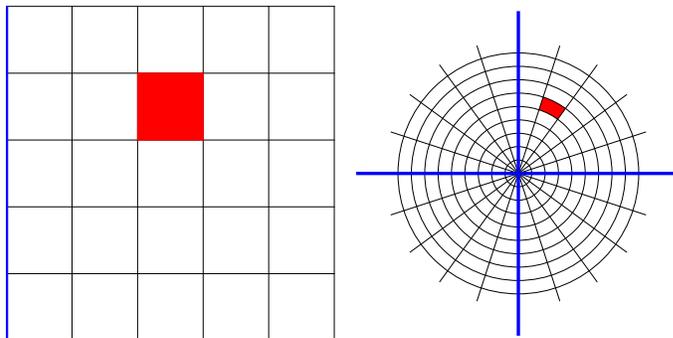
**16.1.** For certain regions, it is better to use a different coordinate system. We first look at integration problems in  $\mathbb{R}^2$ , where polar coordinates  $(x, y) = (r \cos(\theta), r \sin(\theta))$  can help to integrate.

**Definition:** A **polar region** is a planar region bound by a simple closed curve. It is defined in polar coordinates by a curve  $(t, r(t))$  where  $t = \theta$  is the angle. In Cartesian coordinates, the parametrization of the boundary of a polar region is  $\vec{r}(t) = [r(t) \cos(t), r(t) \sin(t)]$ , a **polar graph** like the spiral where the radius  $r(t) = t$  links the distance to the origin with the angle  $t$ .

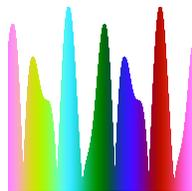
**Theorem:** To integrate in polar coordinates, we evaluate the integral

$$\iint_R f(x, y) \, dx dy = \iint_R f(r \cos(\theta), r \sin(\theta)) r \, dr d\theta .$$

**16.2.** Why do we have to include the factor  $r$ , when we transition to polar coordinates? The reason is that a small rectangle  $R$  with area  $dA = d\theta dr$  in the  $(r, \theta)$  plane is mapped by  $T : (r, \theta) \mapsto (r \cos(\theta), r \sin(\theta))$  to a **sector segment**  $S$  in the  $(x, y)$  plane. It has the area  $r \, d\theta dr$ . We will also see that the parametrization  $\vec{r}(\theta, r) = [r \cos(\theta), r \sin(\theta), 0]$  gives  $|\vec{r}_\theta \times \vec{r}_r| = r$ .



**16.3.** We can now integrate over basic regions in the  $(\theta, r)$ -plane. Examples are **flowers**:  $\{(\theta, r) \mid 0 \leq r \leq f(\theta)\}$ , where  $f(\theta)$  is a non-negative periodic function of  $\theta$ .



A polar region shown in polar coordinates.

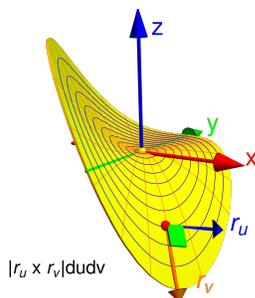


The same region in the  $xy$  coordinate system.

**Theorem:** A surface  $\vec{r}(u, v)$  parametrized on a parameter domain  $R$  has the **surface area**

$$\int \int_R |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \, du \, dv .$$

**16.4. Proof.** The vector  $\vec{r}_u$  is tangent to the grid curve  $u \mapsto \vec{r}(u, v)$  and  $\vec{r}_v$  is tangent to  $v \mapsto \vec{r}(u, v)$ . The two vectors span a **parallelogram** with area  $|\vec{r}_u \times \vec{r}_v|$ . A small rectangle  $[u, u + du] \times [v, v + dv]$  is mapped by  $\vec{r}$  to a parallelogram spanned by  $\vec{r}_u du$  and  $\vec{r}_v dv$  which has the area  $|\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \, du \, dv$ .

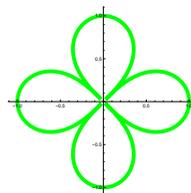
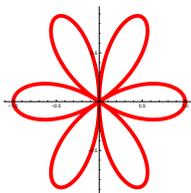
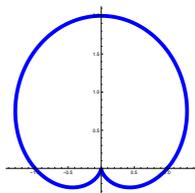


EXAMPLES

**16.5.** The polar graph defined by  $r(\theta) = |\cos(3\theta)|$  belongs to the class of **roses**  $r(t) = |\cos(nt)|$ . Regions enclosed by this graph are also called **rhododenea**. Note that in the literature you often see also situations where  $r(\theta)$  can become negative. We will never allow that as  $r \geq 0$  is a radius and a radius is non-negative in order not to get confused.

**16.6.** The polar curve  $r(\theta) = 1 + \sin(\theta)$  is called a **cardioid**. It looks like a **heart** and belongs to the class of **limaçon** curves  $r(\theta) = 1 + b \sin(\theta)$ .

**16.7.** The polar curve  $r(\theta) = \sqrt{|\cos(2t)|}$  is called a **lemniscate**.



**16.8.** Integrate

$$f(x, y) = x^2 + y^2 + xy$$

over the unit disc. We have  $f(x, y) = f(r \cos(\theta), r \sin(\theta)) = r^2 + r^2 \cos(\theta) \sin(\theta)$  so that  $\iint_R f(x, y) \, dx dy = \int_0^1 \int_0^{2\pi} (r^2 + r^2 \cos(\theta) \sin(\theta)) r \, d\theta dr = 2\pi/4$ .

**16.9.** We have earlier computed area of the disc  $\{x^2 + y^2 \leq 1\}$  using substitution. It is more elegant to do this in polar coordinates:

$$\int_0^{2\pi} \int_0^1 r \, dr d\theta = 2\pi r^2/2|_0^1 = \pi .$$

**16.10.** Integrate the function  $f(x, y) = 1 \{(\theta, r(\theta)) \mid r(\theta) \leq |\cos(3\theta)|\}$ .

$$\int \int_R 1 \, dx dy = \int_0^{2\pi} \int_0^{|\cos(3\theta)|} r \, dr \, d\theta = \int_0^{2\pi} \frac{\cos^2(3\theta)}{2} \, d\theta = \pi/2 .$$

**16.11.** Integrate  $f(x, y) = y\sqrt{x^2 + y^2}$  over the region  $R = \{(x, y) \mid 1 < x^2 + y^2 < 4, y > 0\}$ .

$$\int_1^2 \int_0^\pi r \sin(\theta) r \, r \, d\theta dr = \int_1^2 r^3 \int_0^\pi \sin(\theta) \, d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^\pi \sin(\theta) \, d\theta = 15/2$$

For integration problems, where the region is part of an annular region, or if you see function with terms  $x^2 + y^2$ , always first try to use polar coordinates  $x = r \cos(\theta), y = r \sin(\theta)$ .

**16.12.** The Belgian Biologist **Johan Gielis** defined in 1997 the family of curves given in polar coordinates as

$$r(\phi) = \left( \frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b} \right)^{-1/n_3}$$

This **super-curve** can produce a variety of shapes like circles, squares, triangles or stars. It can also be used to produce “super-shapes”. The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to **describe forms** in biology. <sup>1</sup>



**16.13.** The parametrized surface  $\vec{r}(u, v) = [2u, 3v, 0]$  is part of the xy-plane. The parameter region  $R$  just gets stretched by a factor 2 in the  $x$  coordinate and by a factor 3 in the  $y$  coordinate.  $\vec{r}_u \times \vec{r}_v = [0, 0, 6]$  and we see that the area of  $S = \vec{r}(R)$  is 6 times the area of  $R$ .

<sup>1</sup>Johannes Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003).

**16.14.** The map  $\vec{r}(u, v) = [L \cos(u) \sin(v), L \sin(u) \sin(v), L \cos(v)]$  maps the rectangle  $G = \{0 \leq x \leq 2\pi, 0 \leq y \leq \pi\}$  onto a sphere of radius  $L$ . We compute  $\vec{r}_u \times \vec{r}_v = L^2 \sin(v) \vec{r}(u, v)$ . So,  $|\vec{r}_u \times \vec{r}_v| = L^2 |\sin(v)|$  and  $\int \int_R 1 \, dS = \int_0^{2\pi} \int_0^\pi L^2 \sin(v) \, dv du = 4\pi L^2$ . This is a formula which Archimedes already has derived by seeing it as the surface area of an open cylinder of height  $2L$  and radius  $L$ .

**16.15.** For graphs  $(u, v) \mapsto [u, v, f(u, v)]$ , we have  $\vec{r}_u = [1, 0, f_u(u, v)]$  and  $\vec{r}_v = [0, 1, f_v(u, v)]$ . The cross product  $\vec{r}_u \times \vec{r}_v = [-f_u, -f_v, 1]$  has the length  $\sqrt{1 + f_u^2 + f_v^2}$ . The area of the surface above a region  $R$  is  $\int \int_R \sqrt{1 + f_u^2 + f_v^2} \, dudv$ .

**16.16.** Lets take a surface of revolution  $\vec{r}(u, v) = [v, f(v) \cos(u), f(v) \sin(u)]$  on  $R = [0, 2\pi] \times [a, b]$ . We have  $\vec{r}_u = (0, -f(v) \sin(u), f(v) \cos(u))$ ,  $\vec{r}_v = (1, f'(v) \cos(u), f'(v) \sin(u))$  and  $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v) \cos(u), f(v) \sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$ . The surface area is  $\int \int |\vec{r}_u \times \vec{r}_v| \, dudv = 2\pi \int_a^b |f(v)| \sqrt{1 + f'(v)^2} \, dv$ .

### HOMEWORK

This homework is due on Tuesday, 7/18/2023.

**Problem 16.1:** The **cardioid** has first been described in 1741. Its ultimate fame came because the **main body of the Mandelbrot** set is a Cardioid. Find its area. Its boundary is

$$\vec{r}(t) = \left[ \frac{\cos(t)}{2} - \frac{\cos(2t)}{4}, \frac{\sin(t)}{2} - \frac{\sin(2t)}{4} \right], 0 \leq t \leq 2\pi .$$

**Problem 16.2:** Find  $\int \int_R (x^2 + y^2)^{150} \, dA$ , where  $R$  is the part of the unit disc  $\{x^2 + y^2 \leq 1\}$  for which  $y > x$ .

**Problem 16.3:** Find the area of the region bounded by three curves: first by the polar curve  $r(\theta) = 2\theta$  with  $\theta \in [0, 2\pi]$ , second by the polar curve  $r(\theta) = 3\theta$  with  $\theta \in [0, 2\pi]$  and third by the positive  $x$ -axis?

**Problem 16.4:** Find the average  $\frac{\iint_R f \, dx dy}{\iint_R 1 \, dx dy}$  for  $f(x, y) = 34(x^2 + y^2)$  on the annular region  $R : 1 \leq |(x, y)| \leq 2$ .

**Problem 16.5:** Find the surface area of the part of the paraboloid  $x = y^2 + z^2$  which is inside the cylinder  $y^2 + z^2 \leq 25$ .