

# MULTIVARIABLE CALCULUS

MATH S-21A

## Unit 2: Vectors and dot product

### LECTURE

**2.1.** Two points  $P = (a, b, c)$  and  $Q = (x, y, z)$  in space  $\mathbf{R}^3$  define a **vector**  $\vec{v} = \begin{bmatrix} x - a \\ y - b \\ z - c \end{bmatrix}$ . We write this column vector also as a row vector  $[x - a, y - b, z - c]$  for typographic reasons. As the vector starts at  $P$  to  $Q$  we write  $\vec{v} = \vec{PQ}$ . The real numbers  $p, q, r$  in  $\vec{v} = [p, q, r]$  are called the **components** of  $\vec{v}$ .

**2.2.** Vectors can be attached to any point in space. Two vectors with the same components are considered **equal** as they can be translated into each other. If a vector  $\vec{v}$  starts at the origin  $O = (0, 0, 0)$ , then  $\vec{v} = [p, q, r]$  heads from  $O$  to the point  $P = (p, q, r)$ . One can therefore identify **points**  $P = (a, b, c)$  with **vectors**  $\vec{v} = [a, b, c]$  attached to the origin. For clarity reasons, we often draw an arrow  $\vec{\phantom{v}}$  on top of a vector variable and if  $\vec{v} = \vec{PQ}$  then  $P$  is the “tail” and  $Q$  is the “head” of the vector. To distinguish vectors from points, it is custom to write  $[2, 3, 4]$  for vectors and  $(2, 3, 4)$  for points.<sup>1</sup>

**2.3.**

**Definition:** The **sum** of two vectors is  $\vec{u} + \vec{v} = [u_1, u_2] + [v_1, v_2] = [u_1 + v_1, u_2 + v_2]$ . The **scalar multiple** is  $\lambda\vec{u} = \lambda[u_1, u_2] = [\lambda u_1, \lambda u_2]$ . The **difference**  $\vec{u} - \vec{v}$  can best be seen as the addition of  $\vec{u}$  and  $(-1) \cdot \vec{v}$ .

**Commutativity, associativity, or distributivity** rules for vectors are inherited from the corresponding rules for numbers. Please review these rules yourself if necessary.

**2.4.** The vectors  $\vec{i} = [1, 0, 0]$ ,  $\vec{j} = [0, 1, 0]$ ,  $\vec{k} = [0, 0, 1]$  are called **standard basis vectors**. This has historically grown because the dot and cross product have grown from **quaternions** which are points  $(t, x, y, z)$  in  $\mathbb{R}^4$ , usually written as  $q = t + ix + jy + kz$ .

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<sup>1</sup>We avoid the Stewart notation  $\langle p, q, r \rangle$ , which is harder to read. Programming languages like Python use  $[p, q, r]$ . Mathematica, Perl or C use  $\{p, q, r\}$ . In linear algebra it is useful to distinguish row and column vectors.

## 2.5.

**Definition:** The **length**  $|\vec{v}|$  of a vector  $\vec{v} = \vec{PQ}$  is defined as the distance  $d(P, Q)$  from  $P$  to  $Q$ . A vector of length 1 is called a **unit vector**. If  $\vec{v} \neq \vec{0}$ , then  $\vec{v}/|\vec{v}|$  is called a **direction** of  $\vec{v}$ . The only vector of length 0 is the zero vector  $\vec{0} = [0, 0, 0]$ .

## 2.6.

**Definition:** The **dot product** of two vectors  $\vec{v} = [a, b, c]$  and  $\vec{w} = [p, q, r]$  is defined as  $\vec{v} \cdot \vec{w} = ap + bq + cr$ .

**2.7.** Different notations for the dot product are used in different mathematical fields. While mathematicians write  $\vec{v} \cdot \vec{w}$  or  $(\vec{v}, \vec{w})$  or  $\langle \vec{v}, \vec{w} \rangle$ , the **Dirac notation**  $\langle \vec{v} | \vec{w} \rangle$  is used in quantum mechanics. The **Einstein notation**  $v_i w^i$  or more generally  $g_{ij} v^i w^j$  is used in general relativity. In statistics, it is called the **covariance**  $\text{Cov}[v, w]$  of centered data. The dot product is also called **scalar product** or **inner product**. It could be generalized. Any product  $g(v, w)$  which is bi-linear (linear both in  $v$  and  $w$ ) and satisfies the symmetry  $g(v, w) = g(w, v)$  and  $g(v, v) \geq 0$  and  $g(v, v) = 0$  if and only if  $v = 0$ , can be used as a dot product. An example is  $g(v, w) = 2v_1 w_1 + 3v_2 w_2 + 5v_3 w_3$ .

**2.8.** The dot product determines distances. A bit surprising, distances determine the dot product: **Proof:** Express the length of  $\vec{v}$  as  $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$ . On the other hand, from  $(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2(\vec{v} \cdot \vec{w})$  can be solved for  $\vec{v} \cdot \vec{w}$ :

$$\vec{v} \cdot \vec{w} = (|\vec{v} + \vec{w}|^2 - |\vec{v}|^2 - |\vec{w}|^2)/2.$$

**2.9.** The **Cauchy-Schwarz inequality** is

$$\textbf{Theorem: } |\vec{v} \cdot \vec{w}| \leq |\vec{v}| |\vec{w}|.$$

*Proof.* If  $|\vec{w}| = 0$ , the statement holds because both sides are zero. Otherwise, assume  $|\vec{w}| = 1$  by dividing the equation by  $|\vec{w}|$ . Now plug in  $a = \vec{v} \cdot \vec{w}$  into the equation  $0 \leq (\vec{v} - a\vec{w}) \cdot (\vec{v} - a\vec{w})$  to get  $0 \leq (\vec{v} - (\vec{v} \cdot \vec{w})\vec{w}) \cdot (\vec{v} - (\vec{v} \cdot \vec{w})\vec{w}) = |\vec{v}|^2 + (\vec{v} \cdot \vec{w})^2 - 2(\vec{v} \cdot \vec{w})^2 = |\vec{v}|^2 - (\vec{v} \cdot \vec{w})^2$  which means  $(\vec{v} \cdot \vec{w})^2 \leq |\vec{v}|^2$ .  $\square$

**2.10.** Having established this, it is possible to give, without referring to geometric pictures, a definition of what an **angle** is:

**Definition:** The **angle** between two nonzero vectors  $\vec{v}, \vec{w}$  is defined as the unique  $\alpha \in [0, \pi]$  which satisfies  $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ . Since  $\cos$  maps  $[0, \pi]$  in a 1:1 manner to  $[-1, 1]$ , this is well defined.

**2.11.** The **Al Kashi theorem** gives the third side length  $c$  of a triangle  $ABC$  in terms of the sides  $a = d(A, C)$ ,  $b = d(B, C)$  and  $\alpha$ , the angle located at the vertex  $A$ .

$$\textbf{Theorem: } a^2 + b^2 = c^2 - 2ab \cos(\alpha).$$

*Proof.* Define  $\vec{v} = \vec{AB}$ ,  $\vec{w} = \vec{AC}$ . Because  $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$ , We know  $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$  so that  $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab \cos(\alpha)$ .  $\square$

**2.12.** The **triangle inequality** tells

**Theorem:**  $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$

*Proof.*  $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u} \cdot \vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$ .  $\square$

**Definition:** Two vectors are called **orthogonal** or **perpendicular** if  $\vec{v} \cdot \vec{w} = 0$ . The zero vector  $\vec{0}$  is orthogonal to any vector. For example,  $\vec{v} = [2, 3]$  is orthogonal to  $\vec{w} = [-3, 2]$ .

**2.13.** We can now prove the **Pythagorean theorem**:

**Theorem:** If  $\vec{v}$  and  $\vec{w}$  are orthogonal, then  $|\vec{v} - \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$ .

*Proof.*  $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$ . We usually write this as  $a^2 + b^2 = c^2$ .  $\square$

**2.14.**

**Definition:** The vector  $P(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$  is called the **projection** of  $\vec{v}$  onto  $\vec{w}$ . The **scalar projection**  $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$  is a **signed length** of the vector projection. Its absolute value is the length of the projection of  $\vec{v}$  onto  $\vec{w}$ . The vector  $\vec{b} = \vec{v} - P(\vec{v})$  is a vector orthogonal to the  $\vec{w}$ -direction.

**2.15.** The projection allows to **visualize** the dot product. The absolute value of the dot product is the length of the projection. The dot product is positive if  $\vec{v}$  points more towards to  $\vec{w}$ , it is negative if  $\vec{v}$  points away from it. In the next class, we use the projection to compute distances between various objects.

#### EXAMPLES

**2.16.** For example, with  $\vec{v} = [0, -1, 1]$ ,  $\vec{w} = [1, -1, 0]$ ,  $P(\vec{v}) = [1/2, -1/2, 0]$ . Its length is  $1/\sqrt{2}$ .

**2.17.** The **RGB color space** consists of triples  $\vec{v} = [r, g, b]$  describing the amount of red, green and blue of a **color**. An other coordinate system is the **CMY color space** consisting of triples  $\vec{v} = [c, m, y] = [1 - r, 1 - g, 1 - b]$ , where  $c$  is **cyan**,  $m$  is **magenta** and  $y$  is **yellow**.

**2.18.** In physics, forces and fields  $\vec{F}$  are described by vectors. The **velocity** of a curve  $r(t) = [x(t), y(t), z(t)]$  is a vector attached to the point  $r(t)$ . We will come to this.

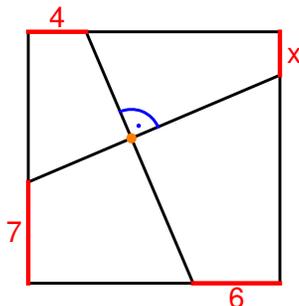
**2.19.** In probability theory, data are described by vectors. One calls them also **random variables**. It is in statistics, where higher dimensional spaces appear. In statistics,  $\vec{v} \cdot \vec{w}$  is the **covariance** and  $\cos(\alpha) = (\vec{v} \cdot \vec{w}) / (|\vec{v}||\vec{w}|)$  is the **correlation**.

HOMEWORK

This homework is due on Tuesday, 6/27/2023.

**Problem 2.1:** a) Find a **unit vector** parallel to  $\vec{x} = \vec{u} + \vec{v} + \vec{w}$  if  $\vec{u} = [4, 3, 1]$  and  $\vec{v} = [2, 4, 1]$  and  $\vec{w} = [0, 0, 4]$ .  
 b) Now find two non-parallel unit vector perpendicular to  $\vec{x}$ .

**Problem 2.2:** Find  $x$  in the following picture about a square. The riddle appeared recently (on June 18, 2023) in a German magazine. Please document how you think about it and also record also attempts which failed. Try to look at the problem from as many point of views as you can.



**Problem 2.3:** Colors are encoded by vectors  $\vec{v} = [\text{red}, \text{green}, \text{blue}]$ . The red, green and blue components of  $\vec{v}$  are all real numbers in the interval  $[0, 1]$ .  
 a) Determine the angle between the colors yellow and magenta.  
 b) What is the vector projection of the magenta-orange mixture  $\vec{x} = (\vec{v} + \vec{w})/2$  onto green  $\vec{y}$ ?

**Problem 2.4:** A rope is wound exactly 8 times around a stick of circumference 1 and length 15. How long is the rope?

**Problem 2.5:** a) Find the angle between the main diagonal of the unit cube and one of the face diagonals. Assume that both diagonals pass through a common vertex.  
 b) Find the vector projection of the main diagonal  $\vec{v} = [1, 1, 1]$  onto the side diagonal  $\vec{w} = [1, 1, 0]$ .  
 c) Find the maximal distance between the 16 points  $(\pm 1, \pm 1, \pm 1, \pm 1)$  of a **tesseract**.