8/3/2023 FINAL PRACTICE 4 **Maths 21a, O.Knill, Summer 2023**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

1) T F The distance d between a point P and a plane through three points A, B, C is given by $d = |\vec{AP} \cdot \vec{BC}|/|\vec{BC}|$.

Solution:
You know the distance formula!

2) T F The set of all points in three dimensional space satisfying $x^2 + y^2 = z^2, x + y + z = 1$ defines a conic section.

Solution:
It is the intersection of a cone with a plane.

3) T F The surface area of a surface parametrized by $\vec{r}(u, v) = [u, v, g(u, v)]$ is given by $\iint_R |g_u \times g_v| \, dudv$.

Solution:
It should be $r_u \times r_v$ and not $g_u \times g_v$.

4) T F Given two curves $\vec{r}_1(t) = [t^3, t^2, t^4]$, and $\vec{r}_2(t) = [t^6, t^4, t^8]$ with curvature functions $k_1(t) = \kappa(r_1(t))$ and $k_2(t) = \kappa(r_2(t))$. Then $k_1(0.5) = k_2(0.5)$.

Solution:
While curvature is independent of the parametrization, we have $r_1(0.5)$ and $r_2(0.5)$ are not the same.

5) T F For any vector field $\vec{F} = [P, Q, R]$, the identity $\text{grad}(\text{div}(\text{curl}(\vec{F})) = \text{div}(\text{grad}(\text{div}(\vec{F})))$ holds.

Solution:
They are different in general. Take $\vec{F} = [0, 0, z^3]$ for example. The left hand side is zero, the right hand side is 6.

6) T F Given two different points A, B with distance 1 in space there exists a suitably parametrized curve $r(\vec{t})$ such that $\int_a^b |\vec{r}\,'(t)| \, dt = 2$.

Solution:
The curve does not have to be the shortest curve, we can make a detour.

7) | T | F | (T highlighted)

If $R = \mathbf{R}^2$ is the entire plane, then the improper integral $\iint_R e^{-x^2-y^2}\, dxdy$ has the value π.

Solution:
The value is π.

8) | T | F | (F highlighted)

The equation $\mathrm{div}(\mathrm{grad}(\mathrm{div}(\mathrm{grad}(f)))) = 0$ is always true for any smooth function f.

Solution:
take $f(x, y, z) = x^4$ for example.

9) | T | F | (T highlighted)

If $\int_a^b \int_c^d f(x, y)\, dxdy = \int_c^d \int_a^b f(x, y) dxdy$ for all a, b, c, d and f is continuous, then $f(x, y) = f(y, x)$.

Solution:
It is not Fubini, but by making the rectangles small, we see it.

10) | T | F | (F highlighted)

If $\vec{v} = \vec{r}(0) = \vec{r}\,'(0)$ is zero, then $\vec{r}(t)$ is zero for all times $t > 0$.

Solution:
No, we can have an acceleration .

11) | T | F | (F highlighted)

A surface S has boundary C with compatible orientation then $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \iint_S \vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v\, dudv$.

Solution:
We would have to take the curl of F.

12) | T | F | (F highlighted)

The equation $y^2 - z^2 + x^2 + 2y = -1$ represents a two-sheeted hyperboloid.

Solution:
Complete the square! It is a cone.

13) | T | F |

$\boxed{T}$ F — A gradient field $\vec{F} = \nabla f$ that is divergence free (=incompressible) has a potential f satisfying the PDE $f_{xx} + f_{yy} + f_{zz} = 0$.

Solution:
Translate what it means.

14) T $\boxed{F}$ — The curvature of the curve $\vec{r}(t) = [25\cos(t^7), 25\sin(t^7)]$ at $t = 1$ is everywhere equal to 5.

Solution:
It is a circle of radius 1/5

15) T $\boxed{F}$ — The distance between lines satisfies the triangle inequality in the sense that $d(L, M) + d(M, K) \geq d(L, K)$ for any three lines K, L, M.

Solution:
We can have L intersect M and M intersect K but L not intersect K.

16) $\boxed{T}$ F — If $\vec{F}$ and $\vec{G}$ are smooth vector fields in $\mathbf{R}^2$ for which the curl is constant 1 everywhere. Then $\vec{F} - \vec{G}$ is a gradient field.

Solution:
It would be the curl of a vector field.

17) T $\boxed{F}$ — Let E is the solid given as the complement of a circle $\{x^2 + y^2 = 1, z = 0\}$ in $\mathbf{R}^3$, then this solid E is simply connected.

Solution:
By definition, not every closed curve can be pulled together to a point.

18) $\boxed{T}$ F — The parametrization $\vec{r}(u, v) = [u^2, u^4 + v^4, v^2]$ is (part of) a paraboloid.

Solution:
Indeed $x^2 + z^2 = y$.

19)	T	F	The flux of the curl of the field $\vec{F} = [x, y, z]$ through the unit sphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ oriented outwards is equal to 4π.

Solution:
By the divergence theorem.

20)	T	F	The flux of the vector field $\vec{F} = [0, 0, 1]$ through the unit disc $\{(x, y, z) \mid x^2 + y^2 \leq 1, z = 0\}$ oriented upwards is equal to the area of the disc.

Solution:
The vector field is perpendicular to the disc, no fluid passes through.

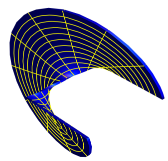
Solution:
One student got 20 points! The minimum was 6.

Problem 2) (10 points) No justifications are necessary.

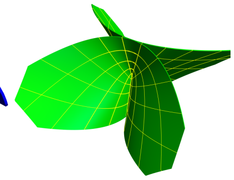
a) (2 points) Match the following surfaces. There is an exact match.

Parametrized surface $\vec{r}(s,t)$	A-C
$[v\cos(u), v\sin(u), u]$	
$[u\cos(v), u, u\sin(v)]$	
$[u^2 v, uv^2, u^2 - v^2]$	

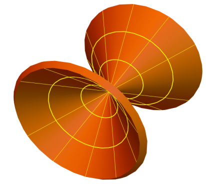
A



B



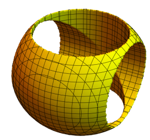
C



b) (2 points) Match the solids. There is an exact match.

Solid	A-C
$x^2+y^2+z^2 < 9, x^2+y^2 > 4, y^2+z^2 > 4$	
$x^2-y^2 < 1, z^2-x^2 < 1, x^2+y^2 < 4$	
$x^2+y^2 < 4, x^2+y^2 > 2, x^2+z^2 > 1$	

A



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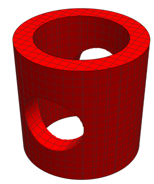

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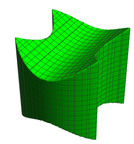
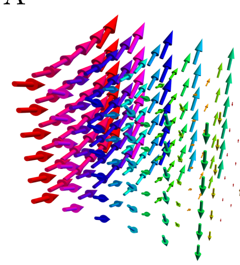


c) (2 points) The figures display vector fields. There is an exact match.

Field	A-C
$\vec{F} = [0, -x\sin(z), x\sin(y)]$	
$\vec{F} = [0, 1-x, x]$	
$\vec{F} = [x, 2y, 3z]$	

A



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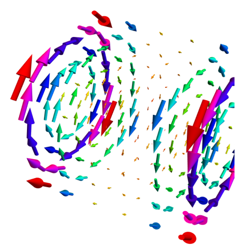

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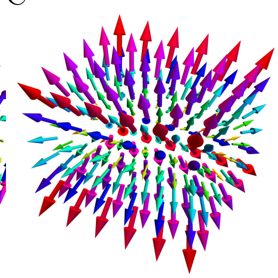


d) (2 points) Recognize partial differential equations!

Equation	A-F
Laplace	
Burger	
Wave	

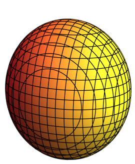
	PDE
A	$X_{tt} = -X_{xx}$
B	$X_t^2 - 1 = X_x^2$
C	$X_t = X_x$

	PDE
D	$X_t = X_x X_{xx}$
E	$X_{tt} = X_{xx}$
F	$-XX_x = X_t$

e) (2 points) Some surfaces

	Enter a letter from A-E each
Pick the one sheeted hyperboloid	
Pick the one elliptic paraboloid	

A



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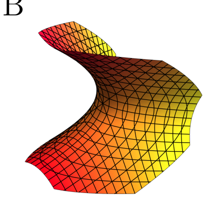

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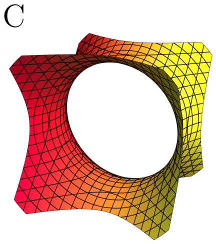
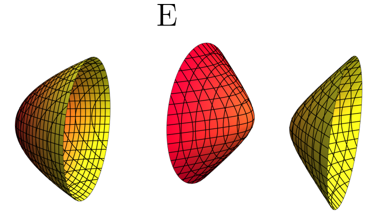


D

E



Solution:
a) ACB
b) ACB
c) BAC
d) AFE
e) CD

Problem 3) (10 points) No justifications necessary

Mathematicians live for ever as they are remembered for ever. But only if we do remember them. So, lets remember them to keep them alive.

a) There was a mathematician who's life inspired the Good Will hunting story. Who was it?

b) Stokes theorem was posed as a calculus exam problem. There was a student taking that exam, who would later become famous. Who was this student?

c) Who proved a theorem about changing the order of integration?

d) Who proved a theorem about changing the order of differentiation?

e) Which two mathematicians are associated to a basic inequality appearing in the dot product? As an alternative, name the inequality.

f) The Newtonian law of gravity as an inverse square force F can be derived from a partial differential equation $\text{div}(F) = 4\pi\sigma$. Who was the mathematician who first looked that way at this?

g) Who was first seeing the principle that at a maximum or minimum, the derivative ∇f must be zero. The person discovered it in one dimensions but the argument is the same in higher dimensions.

h) Which mathematician first looked at extremization problems with constraints?

i) There is a famous fractal in the plane. We saw how to dive deep into the realms up to 10^{-200} and also tried to get the area of that object. Who was the mathematician who found it?

j) There is a partial differential equation used in finance. You have found some solutions in a homework. Name at least one of the persons associated to it.

Bonus: you can regain here one of the lost point in problem 3) by telling, who first discovered the dot and cross product simultaneously by defining a product of 4-tuples.

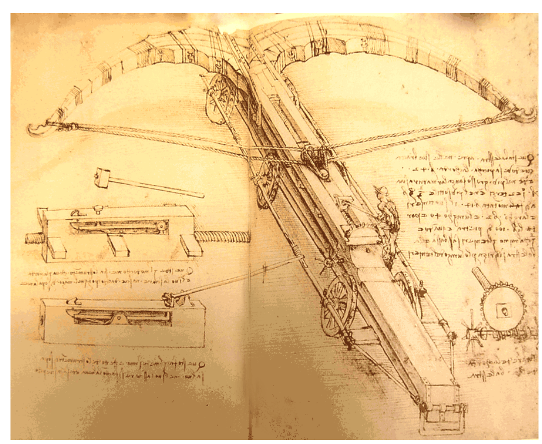
Solution:
Green, Maxwell, Fubini, Clairau, Cauch-Schwarz, Ampere, Fermat, Lagrange, Mandelbrot, Black Scholes an Bonus: Hamilton. This surpise problem was also intended to nag students who skiped classes once in a while. The Maxwell story was only told in class in a review slide show. For the Good will Hunting, we saw a little clip. Fermat was explicitly labeled as Fermat principle both in the notes as well in class. Mandelbrot was mentioned twice in class. Black Sholes was a home work and also appeared again in a slide show. It can not hurt should lost points hurt here: culture and history are important and not just decoration.

Problem 4) (10 points)

Leonardo Da Vinci designed a cross bow. It was heavy and needed to be cocked by mans of a worm gear. We don't believe it has ever been realized. (By the way, as for today, 2022, the best cross bow Scorpyd Aculeus can shoot bows with over 500 km/h, faster then any speed achieved so far by production car.) Assume now we have a cross bow at position $\vec{r}(0) = (2, 0, 50)$ and that the initial velocity is $\vec{r}\,'(0) = (100, 3, 0)$. Assume there is a gravitational and time dependent side wind force

$$\vec{r}\,''(t) = \begin{bmatrix} 1 \\ \sin(t) \\ -10 \end{bmatrix} .$$

Find the trajectory $\vec{r}(t)$ and at which point does it hit the ground $z = 0$?



Solution:
Integrate to get $[t, -\cos(t) - 10t] + [100, 4, 0]$. Note that the 4. The most frequent error was just placing blindly 3. Integrate again to get

$$\vec{r}(t) = [-t^2/2 + 100t + 2, -\sin(t) + 4t, -5t^2 + 50] \ .$$

For z=0, we have $t = \sqrt{10}$. The point is $\vec{r}(\sqrt{10}) = [7 + 1000\sqrt{10}, -\sin(\sqrt{10}) + 4\sqrt{10}, 0]$. An other common mistake was that this final point was not written down. Always read what the problem asks.

Problem 5) (10 points)

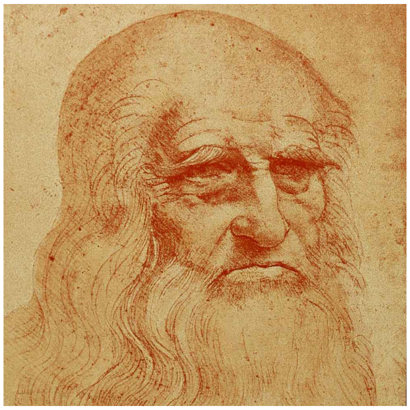
a) (5 points) Find an approximation for the number

$$2.001^2 * 7.003^2 * 9.999^3 \ .$$

by linearizing the function

$$f(x, y, z) = x^2 y^2 z^3$$

at $(2, 7, 10)$.



b) (5 points) What is the equation $ax + by + cz = d$ of the tangent plane to the surface $f(x, y, z) = 196000 = 2^2 * 7^2 * 10^3$ at the point $(x_0, y_0, z_0) = (2, 7, 10)$?

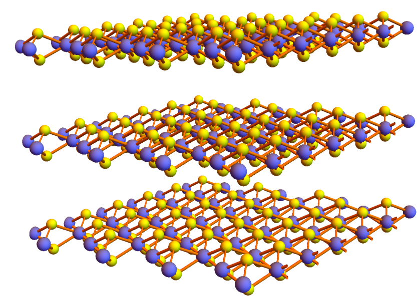
Solution:
a) The gradient is $\nabla f(x, y, z) = [2xy^2z^3, 2x^2yz^3, 3x^2y^2z^2] = xyz^2[2yz, 2xz, 3xy]$. At the point $(2, 7, 10)$ this is $2 \cdot 7 \cdot 100[140, 40, 42] = [a, b, c] = [196K, 56K, 58.8K]$. Now

$$L(x, y, z) = 196K + 196K0.001 + 56K0.003 - 58.5K0.001 = 196305.2 \ .$$

The actual value is 196305.349692816825....
b) The equation is $140x + 40y + 42z = 140 * 2 + 40 * 7 + 42 * 10 = 980$. This was often written ouit also as $196000x + 56000y + 58800z = 1372000$.

Problem 6) (10 points)

Molybdenum disulfide MoS_2 consists of molybdenum Mo and sulfur S. Crystalized molybdenite can be found for example in the Swiss mountains, like in the Baltschieder region. The crystal layer structure is pretty cool. Similarly as graphite, the planes are held together by van der Waals forces. As it has low friction and is added as a solid lubricant. Assume one of the layers is the plane



$$x + 2y + z = 4$$

and on the next layer is an atom with coordinate $P = (3, 4, 5)$.

a) (7 points) What is the distance of P to the plane?

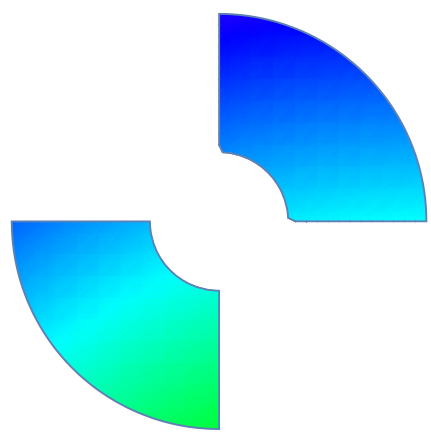
b) (3 points) Please parametrize a line perpendicular to the plane passing through P.

Solution:
a) This is a standard distance problem. We have $\vec{n} = [1, 2, 1]$. Take a point on the plane linke $Q = (4, 0, 0)$. Now the distance is $|\vec{PQ} \cdot n|/\sqrt{n} = |[1, -4, 5] \cdot [1, 2, 1]|/\sqrt{6} = 12/\sqrt{6}$.
b) We have

$$\vec{r}(t) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} .$$

Problem 7) (10 points)

a) (5 points) Evaluate the double integral



$$\int_0^4 \int_0^{x^2} \frac{y^4}{4 - \sqrt{y}} \, dydx$$

b) (5 points) Find the moment of inertia

$$\iint_G x^2 + y^2 \, dxdy$$

for the region $G = \{xy > 0, \;\; 4 < x^2 + y^2 < 9\}$.

Solution:
a) This is a case for a change of integration order. Make a picture. We get

$$\int_0^\infty \int_{\sqrt{y}}^4 \frac{y^4}{4-\sqrt{y}}\, dxdy = \int_0^{16} y^4\, dy = 16^5/5 \; .$$

b) This is a case for polar coordinates. By symmetry, we can integrate over the first quadrant only and muliply by 2. The result is (don't forget the integration factor r, which was also here the most common mistake):

$$2\int_0^{\pi/2} r^2 r\; drd\theta \; = \frac{65\pi}{4} \; .$$

Problem 8) (10 points)

In order to optimize a parachute of Da Vinci, we classify the critical points of the function

$$f(x,y) = 4x^2y + 2x^2 + y^2 \; .$$

The first part is related to the volume, the second part is related to residual volume dragged along during the fall. Don't worry about the derivation of the function $f(x,y)$. It is a Da Vinci thing. So:

a) (8 points) Classify the critical points.

b) (2 points) Decide whether any of the points is a global maximum or global minimum.

Solution:
a) This is a standard extremziation problem. We want to find the points where the gradient $\nabla f(x,y) = [8xy + 4x, 4x^2 + 2y]$ is zero. For the first equation there are the possibilities $x = 0$ or $y = -1/2$. From the second equation we get so the x values $x = \pm 1/4 = \pm 1/2$. The three critical points are $(0,0), (1/2,-1/2)$ and $(-1/2,1/2)$. We compute $f_{xx} = 8y_2$ and $f_{yy} = 2$ and $f_{xy} = 8x$. Then compute $D = f_{xx}f_{yy} - f_{xy}^2$ at each point

Point	D	f_{xx}	Nature	Value
$(-\frac{1}{2},-\frac{1}{2})$	-16	0	saddle	$\frac{1}{4}$
$(0,0)$	8	4	minimum	0
$(\frac{1}{2},-\frac{1}{2})$	-16	0	saddle	$\frac{1}{4}$

b) If we put $y = x$, we have the function $4x^3 + 3x^2$ which grows indefinitely for $x \to \infty$, both to infinity ($x \to \infty$) and to minus infinity ($x \to -\infty$).

Problem 9) (10 points)

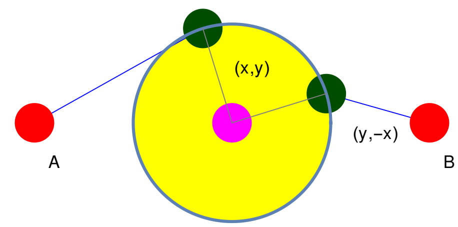
Motivated by a drawing of Da Vinci, we look at the following problem: a wheel of radius 1 is attached by rubber bands to two points $A = (-2, 0)$ and $B = (2, 0)$. The point (x, y) connects to A and $(y, -x)$ to $(2, 0)$. The point (x, y) is constrained to

$$g(x, y) = x^2 + y^2 = 1 \ .$$

The wheel will settle at the position, for which the potential energy $f(x, y) = [(x + 2)^2 + y^2 + (y - 2)^2 + x^2]/2$ which is

$$f(x, y) = x^2 + y^2 + 2x - 2y + 4$$

is minimal. Find that position using the method of Lagrange multipliers.



Solution:
The **Lagrange equations** are

$$\begin{aligned} 2x + 2 &= \lambda(2x) \\ 2y - 2 &= \lambda(2y) \\ x^2 + y^2 &= 1 \end{aligned}$$

Eliminating λ gives $y = -x$. Plugging into the third equation gives $x^2 + (-x)^2 = 1$ so that $(x, y) = (-1/\sqrt{2}, 1/\sqrt{2})$ and $(x, y) = (1/\sqrt{2}, -1/\sqrt{2})$. Evaluating the function f on the two points shows that $(-1/\sqrt{2}, 1/\sqrt{2})$ is the minimum.

Problem 10) (10 points)

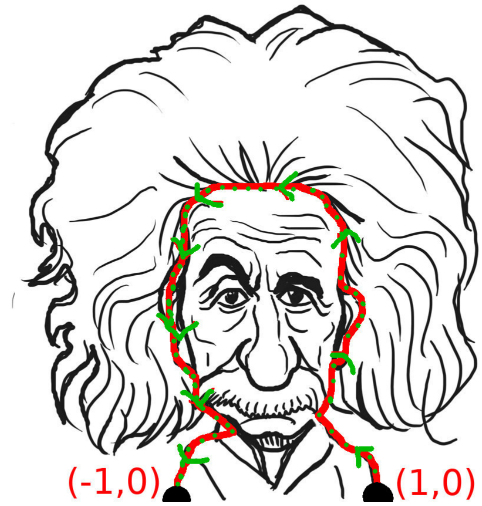
What is the line integral

$$\int_0^1 \vec{F}(r(t)) \cdot \vec{r}\,'(t)\, dt$$

along the **Einstein curve** shown in the picture? The curve goes from $\vec{r}(0) = (1,0)$ to $\vec{r}(1) = (-1,0)$. The vector field is

$$\vec{F}(x,y) = \begin{bmatrix} 4x^3 + y + y^2 \\ 1 + x + 2xy \end{bmatrix} .$$

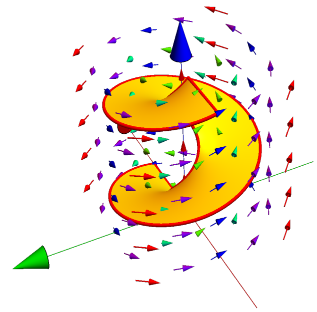
Don't ask for the formula of the Einstein curve. Only Einstein knows.



Solution:
The vector field is a gradient field with potential $f(x,y) = x^4 + xy + y^2x + y$. By the **fundamental theorem of line integrals**, we have

$$\int_0^{2\pi} \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = f(-1,0) - f(1,0) = 1 - 1 = 0$$

Problem 11) (10 points)



We build a model of the **Da Vinci helicopter**. The helicopter blade is a helix S parametrized as

$$\vec{r}(u,v) = \begin{bmatrix} u\cos(v) \\ u\sin(v) \\ v \end{bmatrix}$$

with $0 \leq v \leq 3\pi$ and $1 \leq u \leq 5$. Its boundary C consists of 4 parts. A path going radially out, then the helix up, going radially in and then along the axes down. Let $\vec{F}$ be the vector field

$$\vec{F}(x,y,z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix} .$$

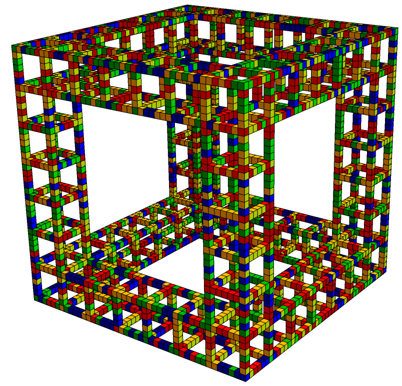
Find the line integral of $\vec{F}$ along the curve C. The curve is oriented so that it is compatible with the surface orientation, which is the orientation given by the parametriation.

Solution:
This is a problem for Stokes theorem. We compute the flux of the field $\text{curl}(\vec{F})$ through the surface. As usual, the most frequently done mistake done here was to compute the flux of $\vec{F}$ and not the flux of the curl. The curl of $\vec{F}$ is $[0,0,2]$. Now compute $\vec{r}_u \times \vec{r}_v = [-\sin(v), -\cos(v), u]$. So, $\text{curl}(\vec{F})(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = 2u$. We have now to integrate this over the domain

$$\int_0^{3\pi} \int_1^5 2u \; dudv = 72\pi .$$

Problem 12) (10 points)

We love the **Menger sponge**. For an exhibit hall, we build a variant, where a cube is divided into 7 parts and the middle 5×5 cylinders are cut out, we end up with a fractal of dimension $\log(68)/\log(7) = 2.1684....$ The second iteration E shown in the picture consists of $68^2 = 4624$ cubes of side length 1. What is the flux of the vector field



$$\vec{F}(x,y,z) = \begin{bmatrix} 4x + e^{\cos(y)} \\ z^5 - y \\ y^5 - z \end{bmatrix}$$

through the boundary S of the solid E assuming as usual that the surface S is oriented outwards?

Solution:
The divergence of $\vec{F}$ is constant and equal to 2. By the **divergence theorem**, the result is 2 times the volume of the solid E. Which is $2 * 4624 = 9248$.

Problem 13) (10 points)

The **"Easy-going region"** D enclosed by the curve

$$\vec{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) - \cos(t) \end{bmatrix}$$

with $0 \leq t \leq 2\pi$ is called the "laidback disk". You can just call it the "Dude" or "His Dudeness" if you are not into that whole brevity thing. What is the area of the dude D?

Solution:
This is a problem for **Green's theorem**. Take the vector field $\vec{F} = [0, x]$ as usual. Then compute the line integral of $\vec{F}$ along the boundary.

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^{2\pi} \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) + \sin(t) \end{bmatrix} dt$$

This is $\int_0^{2\pi} \cos^2(t) + \sin(t)\cos(t)\; dt = \int_0^{2\pi} (1 + \cos(2t))/2 + \sin(2t)/2\; dt = \pi$.